$U \subseteq \mathbb{R}^3$ open, $F : U \to \mathbb{R}^3$ a smooth vector field

$$F(x,y,z) = \left(f_1(X), f_2(X), f_3(X)\right)$$

where $X = (x,y,z)$.

The <u>divergence</u> is

$$\operatorname{div} F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

This is the natural extension of the div we saw previously.

Symbolically, one can write

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = (D_1, D_2, D_3)$$

and say

$$\operatorname{div} F = \nabla \cdot F = D_1 f_1 + D_2 f_2 + D_3 f_3.$$

(<u>scalar-valued</u>)

$E_1 = (1,0,0)$
$E_2 = (0,1,0)$
$E_3 = (0,0,1)$

The <u>curl</u> of F is (symbolically)

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} E_1 & E_2 & E_3 \\ D_1 & D_2 & D_3 \\ f_1 & f_2 & f_3 \end{vmatrix}$$

(vector-valued)

<u>Note:</u> $(x,y,z) = xE_1 + yE_2 + zE_3$

$$= \left(D_2 f_3 - D_3 f_2, -D_1 f_3 + D_3 f_1, D_1 f_2 - D_2 f_1\right)$$

So curl F is another vector field.

Ex: $F(x,y,z) = (\sin xy,\ e^{xz},\ 2x + yz^4)$

$$(\operatorname{div} F)(x,y,z) = \underbrace{y\cos xy + 0 + 4yz^3}_{\text{scalar-valued fcn}}$$

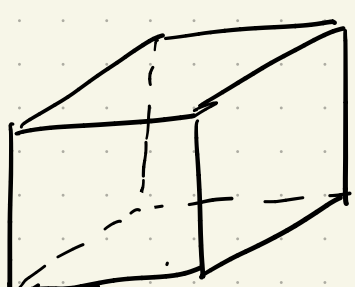
$(\operatorname{curl} F)(x,y,z)$

$$= \begin{vmatrix} E_1 & E_2 & E_3 \\ D_1 & D_2 & D_3 \\ \sin xy & e^{xz} & 2x+yz^4 \end{vmatrix}$$

$$= \underbrace{\left(z^4 - xe^{xz},\ -2,\ ze^{xz} - x\cos xy\right)}_{\text{vector field}}$$

Divergence Thm in $\mathbb{R}^3$

$U \subseteq \mathbb{R}^3$ a region whose boundary is a closed (orientable) surface



closed

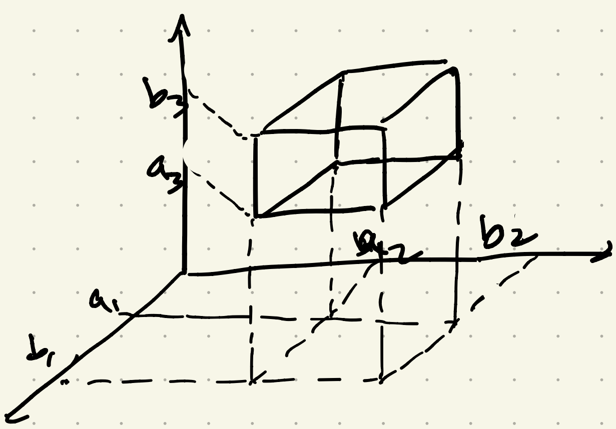
closed if you include top face

Divergence Theorem:

$U \subseteq \mathbb{R}^3$ region whose boundary is a smooth surface S. F defined on a neighborhood of U and S. $\vec{n}$ outward normal.

$$\iint_S F \cdot \vec{n}\, d\sigma = \iiint_U \operatorname{div} F \, dV$$

show this for a box:



$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

The front face S_1 can be parametrized as

$$X_1(y,z) = (b_1, y, z)$$

$$a_2 \leq y \leq b_2$$
$$a_3 \leq z \leq b_3$$

$$\frac{\partial x_1}{\partial y} = (0,1,0), \quad \frac{\partial x_2}{\partial z} = (0,0,1)$$

$\vec{n}_1 = (1,0,0) \longrightarrow$ can compute this from the parametrization or just stare at the picture.

If $F = (f_1, f_2, f_3)$, then

$$\iint_{S_1} F \cdot \vec{n}\, d\sigma = \int_{a_3}^{b_3}\int_{a_2}^{b_2} f_1(b_1, y, z)\, dy\, dz$$

The back face S_2 can be parametrized

$$X(y,z) = (a_1, y, z), \quad n_2 = -(1,0,0)$$

$$\iint_{S_2} F\cdot\vec{n}\, d\sigma = \int_{a_3}^{b_3}\int_{a_2}^{b_2} -f_1(a_1,y,z)\,dy\,dz$$

so $$\iint_{S_1} F\cdot n\, d\sigma + \iint_{S_2} F\cdot n\, d\sigma = \int_{a_3}^{b_3}\int_{a_2}^{b_2}\left(f_1(b_1,y,z) - f_1(a_1,y,z)\right) dy\,dz$$

$$= \int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} D_1 f_1(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_U D_1 f_1\, dV.$$

Pairing off the other 4 sides will give

$$\iiint_U D_2 f_2\, dV = \iint_{\text{side faces}} F\cdot\vec{n}\, d\sigma$$

$$\iiint_U D_3 f_3 = \iint_{\substack{\text{top +}\\ \text{bottom}\\ \text{faces}}} F\cdot\vec{n}\, d\sigma$$

Adding together gives

$$\iiint_U (D_1 f_1 + D_2 f_2 + D_3 f_3)\, dV$$
$$= \iint_S F \cdot \vec{n}\, d\sigma.$$

So the thm holds for boxes.

Ex: $F(x,y,z) = (x^2, y^2, z^2)$

S = unit cube = $[0,1] \times [0,1] \times [0,1]$

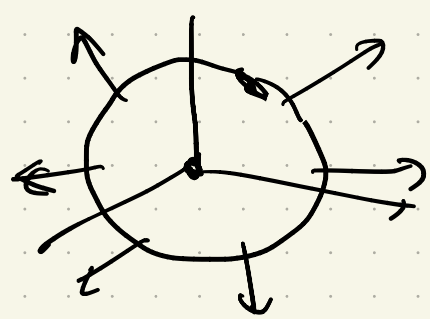
$$\iint_S F \cdot \vec{n}\, d\sigma = \int_0^1\int_0^1\int_0^1 (2x + 2y + 2z)\, dx\, dy\, dz$$

$= \cdots = 3$. The surface integral would require 6 separate integrals.

Ex: $F(x,y,z) = (x,y,z)$, S = sphere of radius a centered at $(0,0,0)$.

B = solid ball

$\operatorname{div} F = 1+1+1 = 3.$



$$\iint_S F \cdot \vec{n}\, d\sigma = \iiint_B \operatorname{div} F\, dV$$
$$= 3 \iiint_B dV = 3 \operatorname{vol}(B)$$
$$= 4\pi a^3.$$

The surface integral is pretty easy to do directly.

$$F(x) = X.$$

$$F \cdot \vec{n} = X \cdot \frac{X}{\|x\|} = \frac{\|x\|^2}{\|x\|} = \|x\|.$$

Since X lies on sphere, $\|x\| = a$.

$$\iint_S F \cdot n \, d\sigma = \iint_S a \, d\sigma = a \iint_S d\sigma = a \, 4\pi a^2 = 4\pi a^3.$$

<u>Corollary:</u> Fix a pt. P. Let $B(t)$ be the solid ball of radius $t > 0$ centered at P. Let $S(t)$ be the boundary of $B(t)$ (i.e. the sphere of radius t). $V(t) := \mathrm{vol}\, B(t)$.

$$(\operatorname{div} F)(P) = \lim_{t \to 0} \frac{1}{V(t)} \iint_{S(t)} F \cdot n \, d\sigma$$

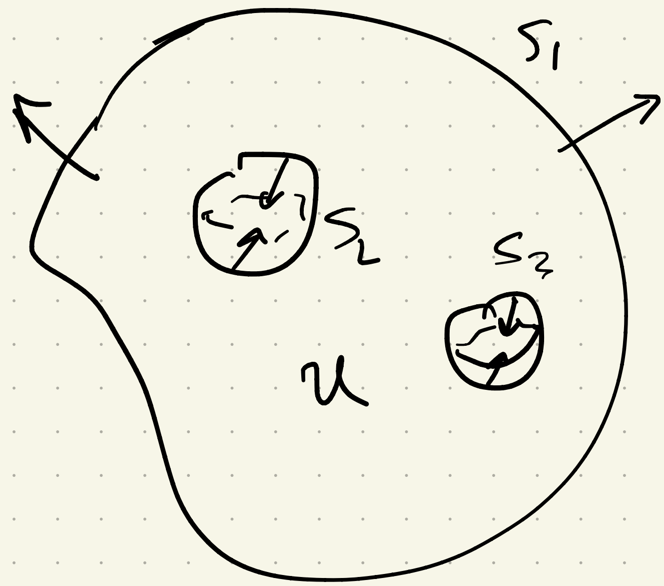
So again, $\operatorname{div} F(P)$ measures net in/out flow in small regions around P.

Divergence Thm (general):

$\mathcal{U}$ open set whose boundary is composed of a finite number of surfaces

$$S = \{S_1, \dots, S_m\},$$

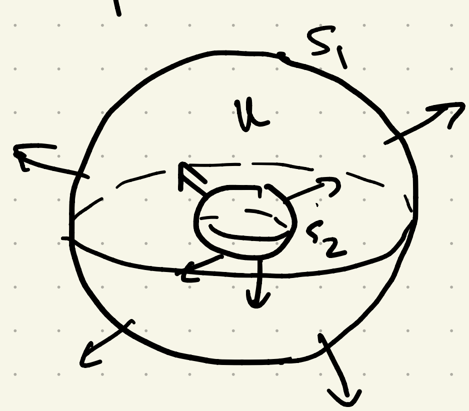
where each S_i is oriented away from $\mathcal{U}$.



Then

$$\iint_S F \cdot n \, d\sigma = \iiint_{\mathcal{U}} \operatorname{div} F \, dV.$$

Ex: Suppose $\operatorname{div} F = 0$ and let $\mathcal{U}$ be the region between two concentric spheres S_1 and S_2



Then $\displaystyle \iint_{S_1} F \cdot n \, d\sigma - \iint_{S_2} F \cdot n \, d\sigma = 0$

since S_2 is oriented the opposite way you need for div. thm

$$\longrightarrow \iint_{S_1} F \cdot n \, d\sigma = \iint_{S_2} F \cdot n \, d\sigma$$

This actually works for any two closed surfaces S_1, S_2 where S_2 is contained in the interior of S_1.

<u>Ex:</u> (Gauss' Law)

$$f(x,y,z) = \frac{q}{4\pi\rho} \qquad \rho = \|X\| = \sqrt{x^2+y^2+z^2}$$

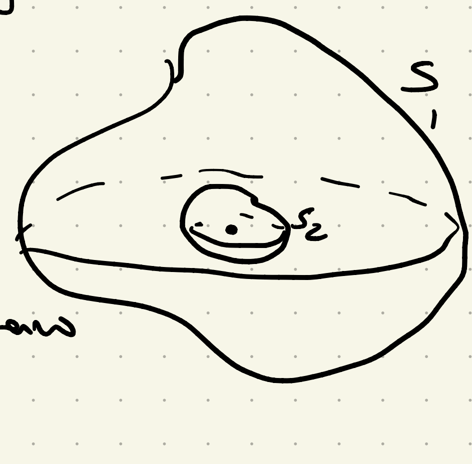
q constant

$$E = -\operatorname{grad} f = \frac{q}{4\pi\rho^3} X.$$

Check that $\operatorname{div} E = 0$.

The previous discussion shows that we can compute the electric flux through any closed surface by deforming it to a sphere.

$$\boxed{\iint_{S_2} E \cdot n \, d\sigma = \frac{q}{4\pi r^2}\left(4\pi r^2\right) = q}$$



$$\frac{q}{4\pi\rho^3} X \cdot \frac{X}{\|X\|} = \frac{q}{4\pi\rho^3} \cdot \frac{\rho^2}{\rho} = \frac{q}{4\pi\rho^2}$$

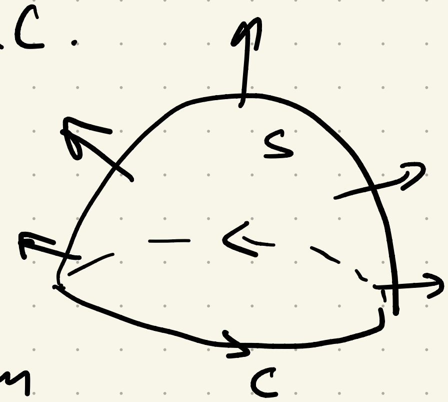
Gauss' Law

Stokes' Thm

Let S be a smooth surface in $\mathbb{R}^3$, bounded by a closed curve C. Assume the surface is oriented and that C has the corresponding induced orientation. Let F be a vec. field.

Then

$$\iint_S (\operatorname{curl} F)\cdot n \, d\sigma = \int_C F \cdot dC .$$



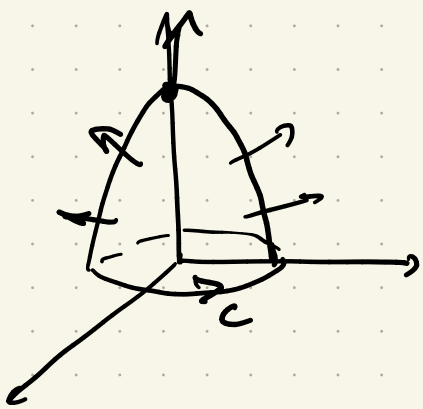
Pf idea: reduce it to Green's thm by a change of variables / transformation.

Note: Green's thm is actually a special case of Stokes', where S lies in the x-y plane so that $\vec{n} = (0,0,1)$, which makes

$$(\operatorname{curl} F)\cdot \vec{n} = \operatorname{rot} F = D_1 f_2 - D_2 f_1$$

Ex: $F(x,y,z) = (z-y,\ x+z,\ -(x+y))$

$z = 4-x^2-y^2,\quad 0 \le z \le 4$



$C(t) = (2\cos t, 2\sin t, 0)$

$C'(t) = (-2\sin t, 2\cos t, 0)$

$0 \le t \le 2\pi$

$F \cdot dC$

$= (0 - 2\sin t,\ 2\cos t,\ -2\cos t - 2\sin t)$

$\cdot\ (-2\sin t, 2\cos t, 0)\, dt$

$= (4\sin^2 t + 4\cos^2 t)\, dt = 4\, dt$

$\int_0^{2\pi} F \cdot dC = \int_0^{2\pi} 4\, dt = 8\pi.$

$$\text{curl } F = \begin{vmatrix} E_1 & E_2 & E_3 \\ D_1 & D_2 & D_3 \\ z-y & x+z & -x-y \end{vmatrix} = (-2, 2, 2)$$

$X(x,y) = (x, y, 4-x^2-y^2),\quad x^2+y^2 \le 4$

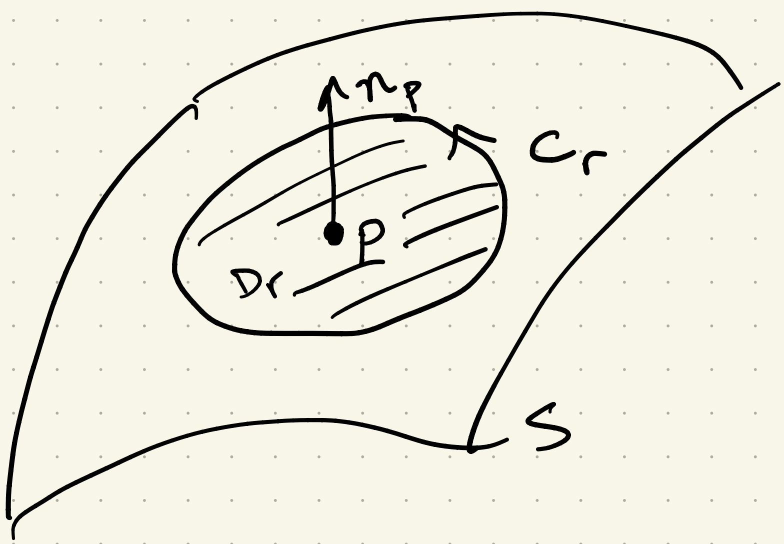
$N(x,y) = \frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y} = (2x, 2y, 1)$

$(\text{curl } F) \cdot N = -4x + 4y + 2$

So $\displaystyle\iint_S \text{curl}\, F \cdot n\, d\sigma = \iint_{x^2+y^2\le 4} \text{curl}\, F \cdot N\, dx\, dy$

$$= \int_0^{2\pi}\int_0^2 (-4r\cos\theta + 4r\sin\theta + 2)\, r\, dr\, d\theta$$

$$= 8\pi.$$



<u>Thm:</u> $\displaystyle\left(\text{curl}\, F(P)\right)\cdot \vec{n}_p = \lim_{r\to 0} \frac{1}{A(r)} \int_{C_r} F \cdot dC$

<u>Pf:</u> similar to 2-dim case.

So curl is a measure of rotation or circulation of F around P.