Curve Integrals

Recall (or accept) from physics that the work (which has the same units as energy) done by a constant force F over a distance D is $W = FD$. This describes the case of the force pointing in the direction of motion. A slightly more general equation is $W = F \cdot D$, where F is the force vector and D is the displacement vector (imagine pushing a box). But this equation still assumes a straight-line displacement and constant force in a fixed direction. What if our trajectory is a curve $C(t)$ and the force is a vector quantity $F(X)$ that depends on position?

If one zooms in close enough on a continuous vector field, it looks constant, and similarly a curve will look like a straight line segment. The work done by the force on a small time interval $(t, t + \Delta t)$ can then be approximated as

$$F(C(t)) \cdot (C(t + \Delta t) - C(t)).$$

We can rewrite this as

$$F(C(t)) \cdot \frac{C(t + \Delta t) - C(t)}{\Delta t} \Delta t.$$

If we add up these small bits of work and let $\Delta t \to 0$, we end up with an integral.

Thus we define the **integral of F along C** from time a to time b as

$$\int_C F = \int_a^b F(C(t)) \cdot \frac{dC}{dt} dt.$$

Example. $F(x, y) = (x^2 y, y^3)$. Find the integral along the straight line from $(0, 0)$ to $(1, 1)$.

We take $C(t) = (t, t)$, $0 \le t \le 1$. $C'(t) = (1, 1)$. Then

$$F(C(t)) = (t^3, t^3).$$

Our integral is then

$$\int_0^1 (t^3, t^3) \cdot (1, 1) dt = \int_0^1 2t^3 dt = 1/2.$$

In 2-space, if we write $F = (f, g)$, $C(t) = (x(t), y(t))$, then the curve integral can be expressed

$$\int_C F = \int_C f dx + g dy.$$

Symbolically, the expression $f dx + g dy = (f, g) \cdot (dx, dy)$. So one can write

$$\int_C F = \int_a^b \left[f(x(t), y(t)) \frac{dx}{dt} + g(x(t), y(t)) \frac{dy}{dt} \right] dt.$$

Remark: The curve integral is independent of the particular parametrization you take. That is, if $C_1(t)$ and $C_2(t)$ trace out the same curve but proceed at different rates, the integral of F over either curve will be the same.

Example. Compute the integral of $F(x, y) = (x^2, xy)$ on the parabola $x = y^2$ from $(1, -1)$ to $(1, 1)$.

We can parametrize our curve as $C(t) = (t^2, t)$, $-1 \leq t \leq 1$. The integral is then

$$\int_C F \cdot dC = \int_{-1}^{1} f(C(t)) \cdot C'(t)dt = \int_{-1}^{1} (t^4, t^3) \cdot (2t+1)dt = \int_{-1}^{1} (2t^5 + t^3)dt.$$

Example. Let

$$G(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

Integrate G on the circle of radius 3 centered at the origin from $(3, 0)$ to $(3\sqrt{3}/2, 3/2)$.

We can parametrize the curve C as $C(t) = (3\cos t, 3\sin t)$ where $0 \leq t \leq \pi/6$, so that $C'(t) = 3(-\sin t, \cos t)$. Now,

$$G(C(t)) = \left(\frac{-3\sin t}{9}, \frac{3\cos t}{9} \right) = \frac{1}{3}(-\sin t, \cos t).$$

So the curve integral is

$$\int_0^{\pi/6} G(C(t)) \cdot C'(t)dt = \int_0^{\pi/6} \frac{1}{3}(-\sin t, \cos t) \cdot [3(-\sin t, \cos t)]dt = \pi/6.$$

Notice that $\pi/6$ is also the change in angle of the parametrized particle over the course of its journey. This is not a coincidence.

An Aside on Differential Forms

A function $f(x, y, z)$ has gradient

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

The **total differential** of f is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

You can view this as a purely symbolic thing, perhaps a fancier way of writing the gradient. However, there is something meaningful about this expression. If we think of dx, dy, dz as small changes in x, y, and z, then this expression gives a way of approximating the corresponding change in f. Remember that differentiability means that for H in some small enough neighborhood of the origin, we can write

$$f(X + H) - f(X) = \operatorname{grad} f(X) \cdot H + \|H\| g(H)$$

where $g(H) \to 0$ as $H \to 0$. The left hand side is the change in f, say Δf, going from X to $X + H$. H itself is the change in input, which we could write $H = (\Delta x, \Delta y, \Delta z)$, where we imagine these are small changes in x, y, and z. Dropping the “error” term $\|H\| g(H)$, this reads

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z.$$

Back to Curve Integrals

Given $x = r\cos\theta$ and $y = r\sin\theta$, we can form their total differentials

$$dx = \cos\theta dr - r\sin\theta d\theta,\ dy = \sin\theta dr + r\cos\theta d\theta.$$

This is equivalent to

$$dx = \cos\theta dr - yd\theta,\ dy = \sin\theta dr + xd\theta.$$

We can rewrite this as

$$dx = \frac{x}{\sqrt{x^2+y^2}}dr - yd\theta,\ dy = \frac{y}{\sqrt{x^2+y^2}}dr + xd\theta.$$

In turn, we can say

$$-ydx = \frac{-xy}{\sqrt{x^2+y^2}}dr - y^2d\theta,\ dy = \frac{xy}{\sqrt{x^2+y^2}}dr + x^2d\theta.$$

Finally, adding these two equations together and solving for $d\theta$ yields,

$$d\theta = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

This differential form is telling us that integral of the right hand side above will *always give the change in angle of the position vector over the course of traversing the curve.*

A **path** C is a sequence of curves $C_1, \dots, C_m$ where each C_i is defined on an interval $[a_i, b_i]$ and if we write $P_i = C_i(a_i)$ and $Q_i = C_i(b_i)$, then $P_{i+1} = Q_i$. In other words, one curve ends where the next one starts. The integral along such a path C is defined as

$$\int_C F := \int_{C_1} F + \cdots + \int_{C_m} F.$$

A **closed path** is one such that $Q_m = P_1$ (we close the loop).

Example. Evaluate the integral of $F = (x^2, xy)$ along the closed path that goes along $y = x^2$ from $(0,0)$ to $(1,1)$, then along the line $y = x$ from $(1,1)$ back to $(0,0)$.

We can parametrize this path as two curves C_1 and C_2. Where $C_1(t) = (t, t^2)$, $0 \le t \le 1$ and $C_2(t) = (1-t, 1-t)$, $0 \le t \le 1$. The integral then becomes

$$\int_C F = \int_0^1 (t^2, t^3)\cdot(1, 2t)dt + \int_0^1 ((1-t)^2, (1-t)^2)\cdot(-1,-1)dt = -\frac{1}{3} + \frac{2}{5}.$$

The Reverse Path

For a curve C defined for $a \le t \le b$, the **reverse curve** C^- is defined by

$$C^- = C(a+b-t), a \le t \le b.$$

Lemma. As one might expect, we have

$$\int_{C^-} F = -\int_C F.$$

This comes from using $u = a + b - t$, $du = -dt$.

Example. Integrate $F(x, y) = (x^2, xy)$ along $y = x$ from $(1, 1)$ to $(0, 0)$. We can use the lemma and instead integrate along the reverse curve and flip the sign of the result. So let $C(t) = (t, t)$, $0 \le t \le 1$. Then what we're after is

$$-\int_0^1 2t^2 dt = -2/3.$$

The **reverse path** of $C_1, \dots, C_m$ is $C_m^-, \dots, C_1^-$.

Path Integrals and Potentials

Theorem. Let F be a vector field on an open set U and suppose $F = \operatorname{grad} \varphi$ for some φ on U. Let C be a path from P to Q. Then

$$\int_C F = \varphi(Q) - \varphi(P).$$

In particular, there is no dependence on the path itself, only on the endpoints.

Proof. Let $g(t) = \varphi(C(t))$. Then $g'(t) = \operatorname{grad} \varphi(C(t)) \cdot C'(t)$. Then we have

$$\int_C F = \int_a^b F(C(t)) \cdot C'(t) dt = \int_a^b \operatorname{grad} \varphi(C(t)) \cdot C'(t) dt.$$

This last integral is the integral of $g'(t)$ from a to b, so this becomes

$$\int_C F = \int_a^b g'(t) dt = g(t)\Big|_a^b = \varphi(C(b)) - \varphi(C(b)) = \varphi(Q) - \varphi(P).$$

□

Corollary. The integral around any closed loop of a conservative vector field is 0. This is because such an integral will end up evaluating to $\varphi(P) - \varphi(P) = 0$.

Notice this gives a test: if you can find a closed loop C with $\int_C F \neq 0$, then F is certainly not conservative.

Example. Let

$$F(x, y, z) = (2xy^3 z, 3x^2 y^2 z, x^2 y^3).$$

Then

$$\varphi(x, y, z) = x^2 y^3 z$$

is a potential. Let $P = (1, -1, 2)$ and $Q = (-3, 2, 5)$. Then for any path from P to Q, we have

$$\int_P^Q F = \varphi(Q) - \varphi(P) = 360 - (-2) = 362.$$

When working with a conservative vector field, the notation $\int_P^Q$ is fine since there's no dependence on the particular path.

Example. Integrating

$$G(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

around the unit circle yields a nonzero value, which shows G is not conservative. The computation will be a homework problem.

Example. Let $G(x,y)$ be as above. Compute the integral along the straight segment path going from $(0,0)$ to $(1,1)$ to $(0,1)$.

Dependence of the Integral on the Path

Theorem. Let U be a connected open set and let F be a vector field on U. Assume that for any two points $P, Q \in U$ the integral

$$\int_{P,C}^{Q} F$$

is independent of the path C in U joining P and Q. There there exists a potential function for F on U.

Proof. Fix some point $P_0 \in U$, then define

$$\varphi(X) = \int_{P_0}^{X} F.$$

Write $F = (f_1, \ldots, f_n)$. We must investigate the quantity

$$\frac{\varphi(X+hE_i) - \varphi(X)}{h} = \frac{1}{h} \left[\int_{P_0}^{X+hE_i} F - \int_{P_0}^{X} F \right].$$

We aim to show that this tends to f_i as $h \to 0$. By path independence, we can split the first integral as $\int_{P_0}^{X} + \int_{X}^{P_0+hE_i}$, which gives us

$$\frac{\varphi(X+hE_i) - \varphi(X)}{h} = \frac{1}{h} \left[\int_{X}^{X+hE_i} F \right].$$

We can compute this integral by taking the path

$$C(t) = X + thE_i, \ 0 \le t \le 1.$$

$C'(t) = hE_i$, and $F(C(t)) = F(X + thE_i)$. Notice that $F \cdot E_i = f_i$. The path integral becomes

$$\frac{\varphi(X+hE_i) - \varphi(X)}{h} = \frac{1}{h} \int_0^1 f_i(X + thE_i) h dt.$$

If we change variables as $u = th$, $du = hdt$, we obtain

$$\frac{\varphi(X+hE_i) - \varphi(X)}{h} = \frac{1}{h} \int_0^h f_i(X + uE_i) du.$$

Letting $h \to 0$, the fundamental theorem of calculus tells us the right side tends to $f_i(X)$. $\square$

Theorem. Let U be an open connected set, and let F be a vector field on U. If the integral of F around every closed path in U is 0, then F has a potential.

Proof. Let P, Q be points in U, and let C and D be paths from P to Q in U. Then C followed by D^- is a closed loop, so that

$$\int_C F + \int_{D^-} F = 0,$$

which implies

$$\int_C F = \int_D F.$$

Now we apply the previous theorem. $\square$

We now have a pretty remarkable theorem.

Theorem. Let F be a vector field defined on the plane minus the origin. Write $F = (f, g)$. Assume $D_2 f = D_1 g$. Let C be the counterclockwise unit circle centered at the origin. If $\int_C F = 0$, then F has a potential. Otherwise, writing

$$k = \frac{1}{2\pi} \int_C F,$$

the field

$$F - kG$$

has a potential, where G is our special example.

Proof. First suppose $\int_C F = 0$. For $X \neq 0$, define $\varphi(X)$ to be the integral along the path in the figure. That is, we get to X by first walking along the circle until we are standing at the

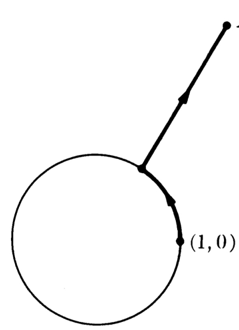

Figure 18

correct angle, and then proceed radially outward/inward to the point X. Our assumption ensures that φ is well-defined. Similar techniques to the previous theorems will show that $\operatorname{grad} \varphi = F$.

In the second case, $F - kG$ has 0 integral around C, so then the first case applies. $\square$