Differentiation

Imagine a bug that moves with constant speed on a circular path of radius r around the origin. The angle of the bug's position vector with the $+x$ axis can be written as

$$\theta = \omega t + a.$$

Assume $a = 0$, so that the bug is on the $+x$ axis at time 0. Then the position vector of the bug is

$$X(t) = (r\cos(\omega t), r\sin(\omega t)).$$

Now imagine the bug lives in $\mathbb{R}^3$ with

$$X(t) = (\cos(t), \sin(t), t).$$

This lifts the circular path into a helix.

In general, a **parametrized curve** $X : I \to \mathbb{R}^n$ is a vector-valued function that maps points from an interval I into n-space. In the examples above, I is the entire real line $\mathbb{R}$ (which we consider to be an interval). We can write $X(t)$ as its individual coordinate functions

$$X(t) = (x_1(t), \ldots, x_n(t)).$$

Just as with ordinary real-valued function, we can take derivatives by looking at the limit

$$\lim_{h\to 0} \frac{X(t+h) - X(t)}{h}.$$

Here, dividing by h really means scaling the vector by $1/h$. Writing out components, this is simply

$$\lim_{h\to 0} \frac{(x_1(t+h) - x_1(t), \ldots, x_n(t+h) - x_n(t))}{h}.$$

If the individual components are all differentiable, we obtain a new vector-valued function

$$X'(t) = (x_1'(t), \ldots, x_n'(t)).$$

$X'(t)$ is called the **derivative** or **velocity** of $X(t)$.

So for the example $X(t) = (\cos(t), \sin(t), t)$, we have

$$X'(t) = (-\sin(t), \cos(t), 1).$$

The velocity is parallel to the direction of instantaneous motion.

Example. Find a parametric equation of the tangent line to the curve $X(t) = (\sin t, \cos t)$ at $t = \pi/3$.

We need two pieces of information: a point on the line, and a direction vector of the line. These are supplied by $X(\pi/3)$ and $X'(\pi/3)$ respectively. The tangent line $L(t)$ can thus be written

$$\begin{aligned} L(s)|_{t=\pi/3} &= X(\pi/3) + sX'(\pi/3) \\ &= \left(\frac{\sqrt{3}}{2} + \frac{1}{2}s,\ \frac{1}{2} - \frac{\sqrt{3}}{2}s \right). \end{aligned}$$

We used the parameter s for the line to avoid confusion with the already defined $X(t)$ above.

The **speed** of the curve $X(t)$, denoted $v(t)$, is defined to be

$$v(t) = \|X'(t)\|.$$

acceleration is the second derivative $X''(t)$.

We note also that differentiation is linear, meaning

$$\frac{d}{dt}\left(X(t) + Y(t)\right) = X'(t) + Y'(t)$$

and

$$\frac{d}{dt}cX(t) = cX'(t).$$

We also have a product rule:

$$\frac{d}{dt}X(t) \cdot Y(t) = X'(t) \cdot Y(t) + X(t) \cdot Y'(t).$$

This follows from applying the ordinary product rule. If $X(t) = (x_1(t), x_2(t))$ and $Y(t) = (y_1(t), y_2(t))$, then

$$\begin{aligned} \frac{d}{dt}X(t) \cdot Y(t) &= \frac{d}{dt}(x_1y_1 + x_2y_2) \\ &= x_1'y_1 + x_1y_1' + x_2'y_2 + x_2y_2' \\ &= x_1'y_1 + x_2'y_2 + x_1y_1' + x_2y_2' \\ &= X'(t) \cdot Y(t) + X(t) \cdot Y'(t). \end{aligned}$$

Of course, this same argument works in dimensions higher than 2.

Lang uses the notation $X(t)^2$ for $X(t) \cdot X(t) = \|X(t)\|^2$. Using this, the above formula has as a particular case

$$\frac{d}{dt}X(t)^2 = 2X(t) \cdot X'(t).$$

Length of Curves

If we integrate the speed $v(t)$ of $X(t)$ from time $t = a$ to $t = b$, we obtain the distance or length traveled by $X(t)$ during the time interval $[a, b]$:

$$\text{length} = \int_a^b v(t)dt.$$

Example. Let $X(t) = (\cos(t), \sin(t))$ describe a particle. What distance does $X(t)$ traverse from $t = 0$ to $t = 1$?

We have $X'(t) = (-\sin(t), \cos(t))$. Then $v(t) = \|X'(t)\| = \sqrt{(-\sin(t))^2 + \cos^2(t)} = 1$. So the distance D is

$$D = \int_0^1 1dt = 1.$$

Note that distance and displacement are not the same thing. In the example above, if we consider the distance traveled from $t = 0$ to $t = 2\pi$, the particle travels a distance of 2π, but the net displacement is 0 since it ends up where it started.

Suppose $X(t) = (x_1(t), x_2(t))$. Then the length integral can be written as

$$\int_a^b \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2} dt.$$

This might seem familiar. In fact, consider now a real-valued function $f(x)$. We can parametrize the graph of f from $x = a$ to $x = b$ as

$$X(t) = (t, f(t)), \;\; a \leq t \leq b.$$

Slotting this into the integral above gives

$$\int_a^b \sqrt{1 + f'(t)^2} dt,$$

which is the arclength formula you may have seen in Calc II.