Calc I Integrals Revisited

Recall the construction of the (Riemann) integral of a function f defined on $[a, b]$. We take a **partition**

$$a = x_1 \leq x_2 \leq \cdots \leq x_m = b$$

of the interval. On each subinterval $[x_i, x_{i+1}]$, we choose a value t_i lying in that subinterval. The corresponding Riemann sum is

$$\sum_{i=1}^{m-1} f(t_i)(x_{i+1} - x_i).$$

The **width** of this partition P is defined to be the largest of the values $x_{i+1} - x_i$. A function is then Riemann integrable if there is some value S such that

$$\sum_P f(t_i)(x_{i+1} - x_i) \to S$$

as $width(P) \to 0$. Notice that this must happen independently of the choice of the t_i's.

Closely related is the Darboux integral. For a given partition P, one defines

$$U(f, P) = \sum_P \sup_{[x_i, x_{i+1}]} f(t)(x_{i+1} - x_i), \;\; L(f, P) = \sum_P \inf_{[x_i, x_{i+1}]} f(t)(x_{i+1} - x_i).$$

If for $\epsilon > 0$ there is some partition P so that $U(f, P) - L(f, P) < \epsilon$, f is said to be Darboux integrable. It turns out that Riemann and Darboux integrability are equivalent.

An important feature of the Riemann integral is that it can deal with a finite number of discontinuities, whether they're jump discontinuities or removable, so long as f is bounded.

Boundedness, Supremum, Infimum

A function f defined on a region R is **bounded** if for some number M, $|f(X)| < M$ for all $X \in \mathbb{R}$.

A set of real numbers A is bounded if it fits inside an interval (a, b). In this case, a is a lower bound on the set, and b is an upper bound. But we can possibly do better. We can slide b to the left, and as long as its still an upper bound, we can keep pushing it. At some point, we'll run into points of A and can't go on. Thus we have the concept of a **supremum** or **least upper bound**. Similarly, by sliding a to the right, we have the notion of an **infimum** or **greatest lower bound**.

Double Integrals

The theory of integration is a bit more technical than the rest of what we've done so far, so the proofs of several theorems are omitted from Lang. It's nevertheless helpful to discuss a few of the fundamental notions involved.

The set $[a, b] \times [c, d]$ consists of those points (x, y) such that $a \leq x \leq b$ and $c \leq x \leq d$. This is a rectangle in the plane.

A **partition** of an interval $[a, b]$ is a sequence of numbers

$$a = x_1 \leq x_2 \leq \cdots \leq x_m = b.$$

This is sometimes denoted $(x_1, x_2, \ldots, x_m)$.

If we partition $[a, b]$ as $(x_1, \ldots, x_m)$ and $[c, d]$ as $(y_1, \ldots, y_n)$, this subdivides the rectangle $R = [a, b] \times [c, d]$ into smaller rectangles.

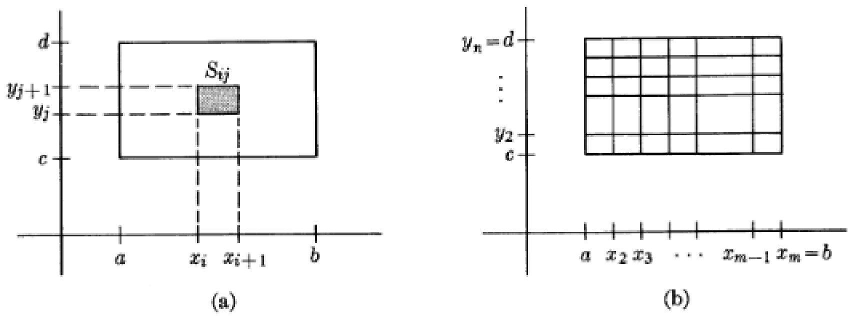

Figure 2

We denote by S_{ij} the subrectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$.

Double integrals are defined very similarly as in the single variable case.

$$U(f, P) = \sum_S (\sup_S f)(\text{Area}(S)), \;\; L(f, P) = \sum_S (\inf_S f)(\text{Area}(S)).$$

The double integral has two nice interpretations, one as a volume, and the other as a mass.

In the same way the Riemann integral on an interval can handle discontinuties as long as they are not too abundant, so too can the Riemann integral on a region.

Theorem. Let R be a rectangle and let f be bounded on R and continous except at possibly at points lying on a finite number of curves. Then f is integrable on R.

A lot of the familiar properties of integrals still work, like additivity and scaling.

Theorem. If $A = A_1 \cup A_2$ where A_1 and A_2 only overlap at possibly a finite number of curves, then

$$\iint_A f = \iint_{A_1} f + \iint_{A_2} f.$$

Also, if A is some smooth curve contained in a rectangle R, and f is zero everywhere in R except possibly at points in A, then

$$\iint_R f = 0.$$