Repeated Integrals

f defined on $[a,b]\times[c,d]$.

Fix x, vary y to form the one variable integral

$$\int_c^d f(x,y)\,dy.$$

As we move x, the value of the integral changes. Hence this is a function of x that we can integrate:

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx,$$ also written

$$\underbrace{\int_a^b \int_c^d f(x,y)\,dy\,dx}_{\text{"repeated integral"}}.$$

Ex 1: Let $f(x,y) = x^2 y$. Compute the repeated integral of f on $[1,2] \times [-3,4]$.

$$\int_1^2 \int_{-3}^4 x^2 y \, dy \, dx$$

$$= \int_1^2 \frac{1}{2} x^2 y^2 \Big|_{y=-3}^{y=4} dx = \int_1^2 \frac{1}{2}(16-9) x^2 \, dx$$

$$= \frac{7}{2} \int_1^2 x^2 dx = \frac{7}{2} \frac{x^3}{3} \Big|_1^2 = \frac{7}{2} \cdot \frac{1}{3} (8-1)$$

$$= \frac{49}{6}.$$

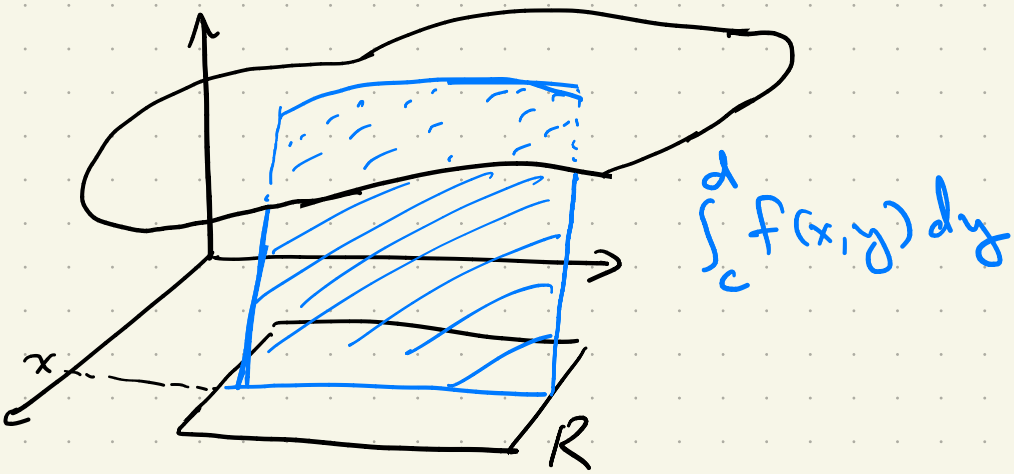
Theorem (Fubini): Let $R = [a,b] \times [c,d]$, and let f be integrable on R. Suppose $\int_c^d f(x,y) \, dy$ exists for each $x \in [a,b]$. Then

$$\iint_R f = \int_a^b \int_c^d f \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy.$$

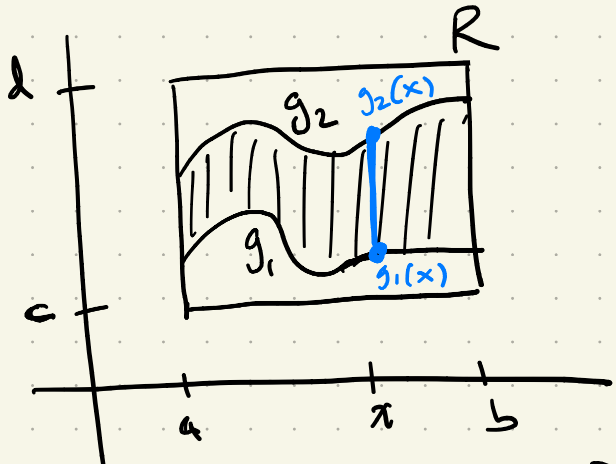
Thus, we can say in the above example that

$$\iint_R x^2 y = \frac{49}{6}.$$

$\int_c^d f(x,y)\,dy$ gives the area of a cross section



Varying x and integrating this area function wrt x then gives a volume.



Suppose we have a region of the form

$$A = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{c} a \le x \le b \\ \text{and} \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$$

Suppose f is cts on A. Define f to be 0 on $R \setminus A$.

Then for a fixed x, the integral $\int_c^d f(x,y)\,dy$ can be written

$$\int_c^d f(x,y)\,dy = \underset{①}{\int_c^{g_1(x)} f(x,y)\,dy} + \underset{②}{\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy} + \underset{③}{\int_{g_2(x)}^{d} f(x,y)\,dy}$$

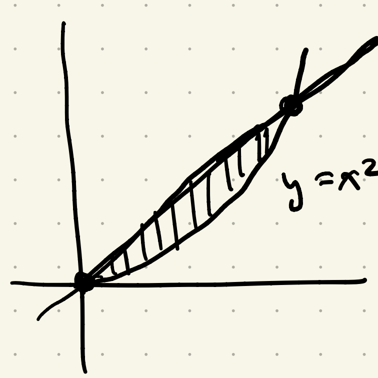
① and ③ are 0 since f is 0 outside the region A. So we have

$$\boxed{\int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx}$$

This is very useful for computing double integrals.

<u>Example</u>: $f(x,y) = x^2 + y^2$

$$A = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} x^2 \le y \le x \\ \text{and } 0 \le x \le 1 \end{array} \right\}$$



$$\iint_A f = \int_0^1 \left[\int_{x^2}^{x} (x^2 + y^2)\,dy \right] dx$$

$$= \int_0^1 \left(x^2 y + y^3/3 \right) \Big|_{y=x^2}^{y=x} dx$$

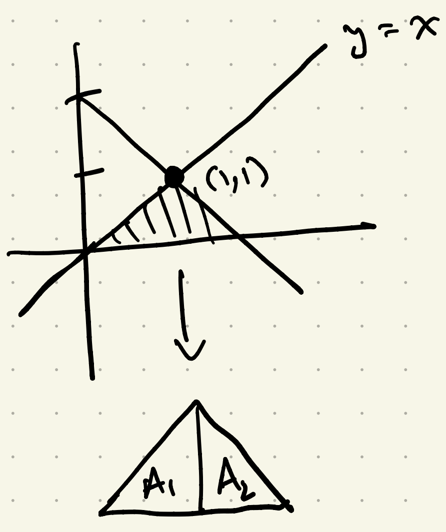
$$= \int_0^1 \left(x^3 + \frac{x^3}{3} - x^4 - \frac{x^6}{3} \right) dx$$

$$= \frac{1}{3}x^4 - \frac{x^5}{5} - \frac{x^7}{21}\Big|_0^1 = \frac{1}{3} - \frac{1}{5} - \frac{1}{21}.$$

If we have a region $A = A_1 \cup A_2$ where A_1, A_2 only overlap on the boundary, then $\iint_A f = \iint_{A_1} f + \iint_{A_2}$, which gives us a way to apply the above ideas to more complicated regions.

<u>Ex:</u> $f(x,y) = 2xy$.

$A =$ region bounded by $y=0$, $y=x$, $x+y=2$.



$$\iint_{A_1} f = \int_0^1\int_0^x 2xy\,dy\,dx$$

$$= \int_0^1 xy^2\Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 x^3\,dx = \frac{1}{4}.$$

$$\iint_{A_2} f = \int_1^2\int_0^{2-x} 2xy\,dy\,dx$$

$$= \int_1^2 xy^2\Big|_{y=0}^{y=2-x} dx = \int_1^2 x(2-x)^2\,dx = 5/12.$$

So $\iint_A f = \frac{1}{4} + \frac{5}{12}$.

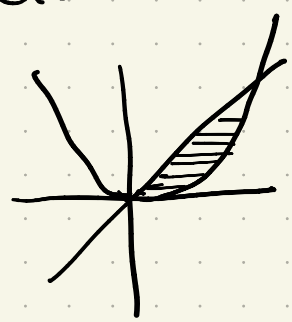
We can also evaluate this integral in the $dxdy$ order.

$$A = \left\{ (x,y) \in \mathbb{R}^2 : \begin{matrix} y \le x \le 2-y \\ 0 \le y \le 1 \end{matrix} \right\}$$

$$\text{so } \iint_A f = \int_0^1 \int_y^{2-y} 2xy \, dx \, dy .$$

Note that $\text{Area}(A) = \iint_A 1 \, dy dx = \iint_A dx dy$.

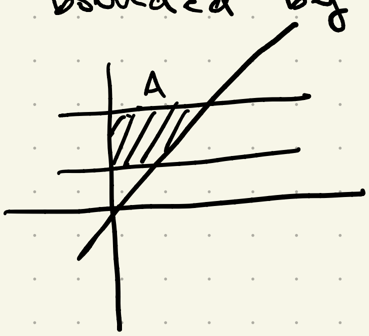
<u>Ex:</u> Find the area of the region between $y = x$ and $y = x^2$



$$\text{Area} = \int_0^1 \int_{x^2}^{x} dy dx = \int_0^1 (x - x^2) dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

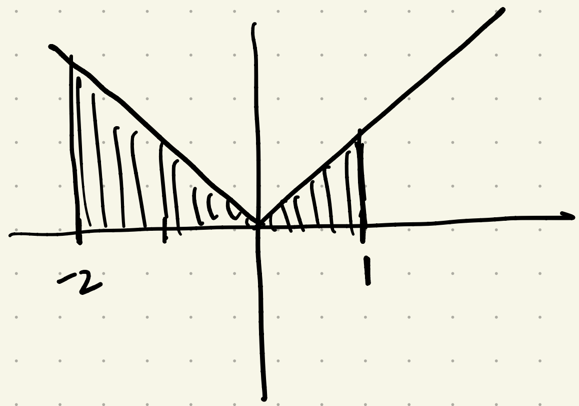
<u>Ex:</u> Find the integral of $f(x,y) = x^2 y^2$ over the region bounded by $y=1$, $y=2$, $x=0$, $x=y$



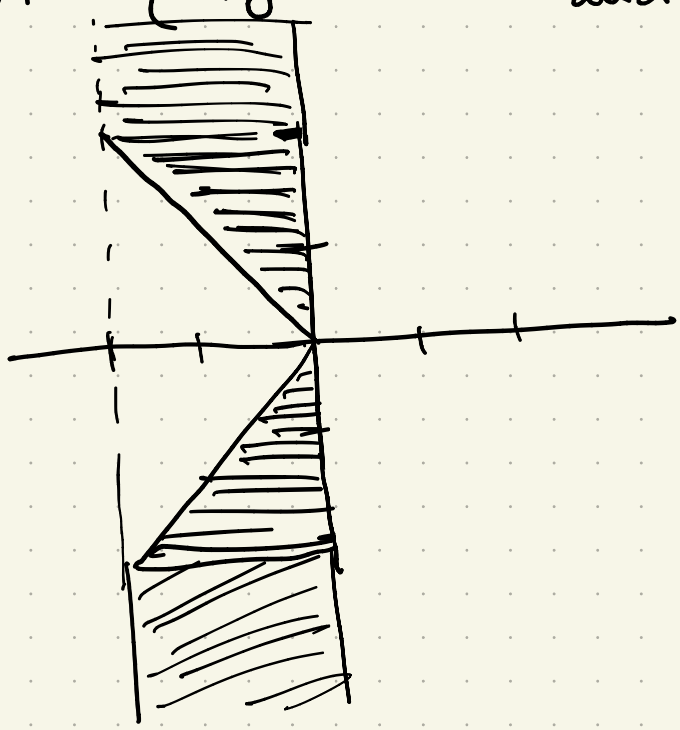
$$\int_1^2 \int_0^y x^2 y^2 \, dx dy = \frac{7}{2}.$$

Ex: Sketch the region

$$A = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} 0 \le y \le |x|, \\ -2 \le x \le 1 \end{array} \right\}$$



Ex: $A = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} |y| \ge |x| \\ \text{and } -2 \le x \le 0 \end{array} \right\}$

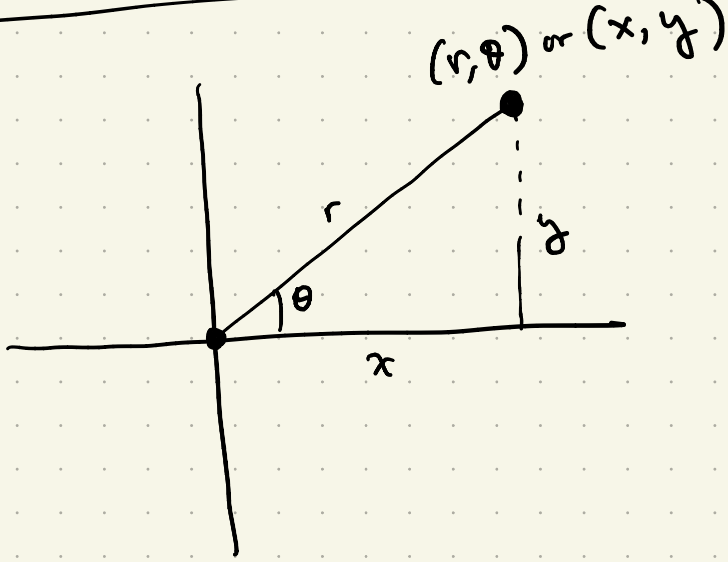


since $x \le 0$ on A, $|x| = -x$, so

$|y| \ge |x| \iff |y| \ge -x$

$\iff x \ge -|y|$

Polar Coordinates



$x = r\cos\theta$

$y = r\sin\theta$.

$$\Downarrow$$

$$r = \sqrt{x^2+y^2}$$

Ex: Find the polar coords of $(1,\sqrt{3})$.

$$r = \sqrt{1+3} = 2$$

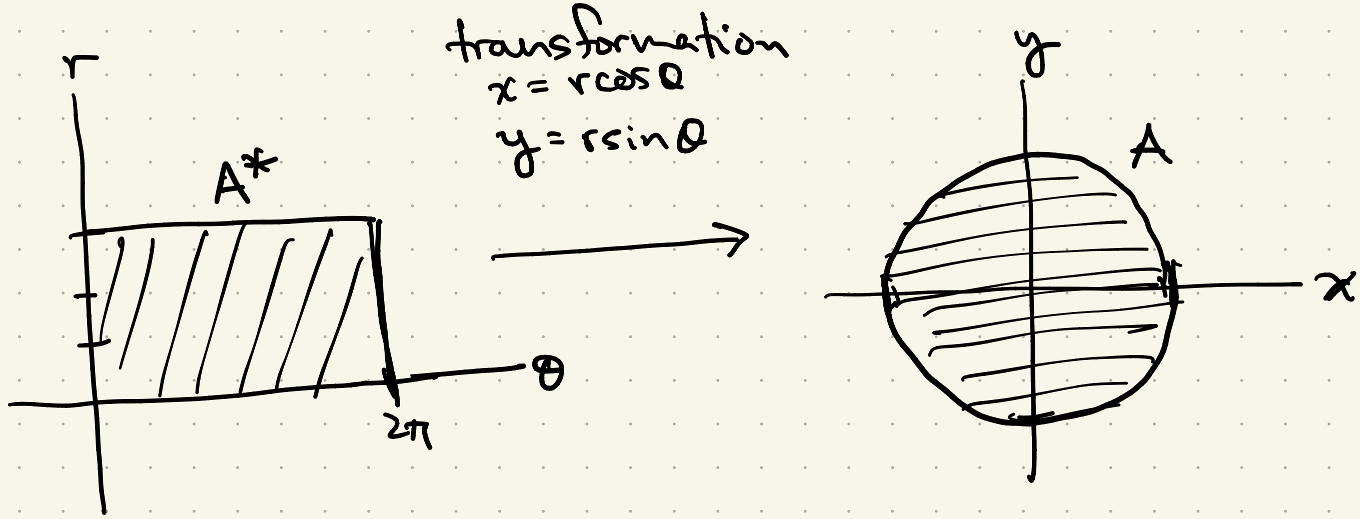
$$\begin{cases} 1 = 2\cos\theta \\ \sqrt{3} = 2\sin\theta \end{cases} \Longrightarrow \begin{cases} \cos\theta = \frac{1}{2} \\ \sin\theta = \frac{\sqrt{3}}{2} \end{cases}$$

$$\Longrightarrow \theta = \frac{\pi}{3}, \text{ so } (r,\theta) = \left(2, \frac{\pi}{3}\right).$$

Note: $(r,\theta) = (r, \theta + 2\pi k)$ for any k.

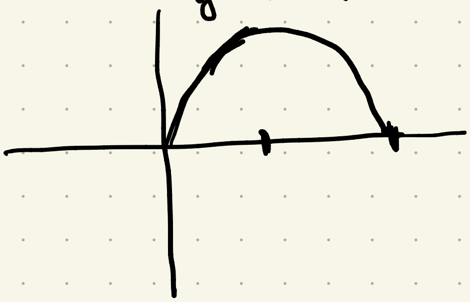
Ex: In polar coordinates, the disk of radius 3 centered at the origin is

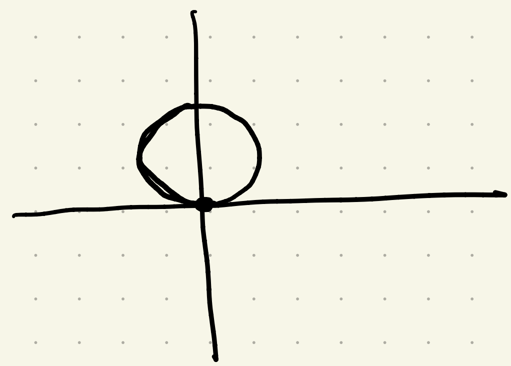
$$0 \le \theta \le 2\pi$$
$$0 \le r \le 3$$



So in the (r, θ) plane, the region is a rectangle.

<u>Ex:</u> graph $r = \sin\theta$ for $\theta \in [0, \pi]$.





$$r = \sin\theta \Rightarrow r^2 = r\sin\theta \Rightarrow x^2 + y^2 = y$$

$$\Rightarrow x^2 + y^2 - y = 0 \Rightarrow x^2 + \left(y - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} = 0$$

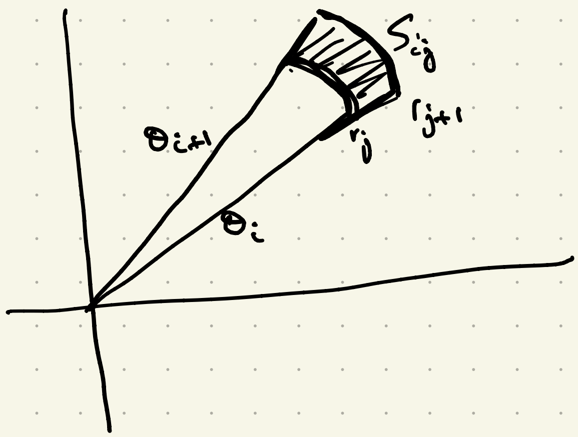
$$\Rightarrow x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}.$$

Consider partitions

$$a = \theta_1 \leq \theta_2 \leq \ldots \leq \theta_n = b$$

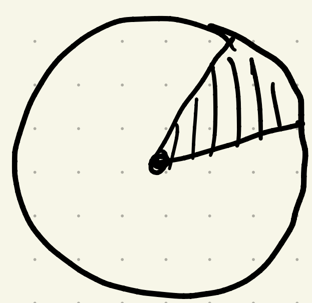
$$c = r_1 \leq r_2 \leq \ldots \leq r_m = d.$$

each pair $[\theta_i, \theta_{i+1}]$, $[r_j, r_{j+1}]$ determines a sectorial piece.



The area of a sector/wedge with angle θ and radius r is

$$\frac{\theta}{2\pi}\pi r^2 = \frac{\theta r^2}{2}$$



so $$\text{Area}(S_{ij}) = \frac{(\theta_{i+1} - \theta_i) r_{j+1}^2}{2} - \frac{(\theta_{i+1} - \theta_i) r_j^2}{2}$$

$$= (\theta_{i+1} - \theta_i) \frac{r_{j+1} + r_j}{2} (r_{j+1} - r_j)$$

writing $\frac{r_{j+1}+r_j}{2} = \bar{r}_j$, we have

$$\text{Area}(S_{ij}) = \bar{r}_j (r_{j+1} - r_j)(\theta_{i+1} - \theta_i).$$

If f is a function on the x-y plane, the corresponding function of (r, θ) is

$$f^*(r,\theta) = f(r\cos\theta, r\sin\theta).$$

Ex: $f(x,y) = 2x^2y \Longrightarrow f^*(r,\theta) = 2r^2\cos^2\theta \, r\sin\theta$
$= 2r^3\cos^2\theta\sin\theta.$

So for a sector $S: a \le \theta \le b,\ c \le r \le d$ in the plane, we can form the Riemann sum

$$\sum_{j=1}^{m-1}\sum_{i=1}^{n-1} f^*(\bar{r}_j, \theta_i)\,\bar{r}_j (r_{j+1} - r_j)(\theta_{i+1} - \theta_i).$$

If we let S^* be the corresponding rectangle in the θ-r plane, it is now reasonable to claim

$$\iint_S f(x,y)\,dy\,dx = \iint_{S^*} f^*(r,\theta)\,r\,dr\,d\theta$$

$$\boxed{dy\,dx = r\,dr\,d\theta}$$