Vector Space

The model example of a vector space is $\mathbb{R}^n$.

- **Def:** A vector space over $\mathbb{R}$ is a set V with an addition $(+)$ operation and scaling operation s.t.

addition:
$u, v \in V$
$u+v \in V$

scaling:
$\lambda \in \mathbb{R}$,
$v \in V$
$\lambda v \in V$

  - $u+v = v+u$ for all $u, v \in V$
  - $(u+v)+w = u+(v+w)$ and
    $(ab)v = a(bv)$ for all $u, v, w \in V$
    $a, b \in \mathbb{R}$.
  - There exists $0 \in V$ s.t.
    $v + 0 = v$ for all $v \in V$.
  - $\forall v \in V$, there is $w \in V$ s.t.
    $v + w = 0$.
  - $1v = v \quad \forall v \in V$.
  - $a(u+v) = au + av \quad a, b \in \mathbb{R}$
    $(a+b)v = av + bv \quad u, v \in V$.

Elements of V are called vectors or points.

Can replace $\mathbb{R}$ w/ $\mathbb{C}$ (or any field)

We immediately see $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

If S is a set, then $\mathbb{R}^S$ denotes the set of functions from S to $\mathbb{R}$. This a v.s. over $\mathbb{R}$.

A vector space has a unique additive identity.

<u>Pf:</u> If $0, 0'$ are additive identities, then

$$0' = 0' + 0 = 0 + 0' = 0.$$

Every element has a unique additive inverse

Pf: Suppose w, w' are additive inverses of v. Then $w = w + 0 = w + (v + w')$

$$= (w+v) + w' = 0 + w' = w'.$$

we denote the additive inverse of v by $-v$.

$w - v$ is defined as $w + (-v)$.

$0v = 0$

$w + w = w$
$\Rightarrow w = 0$

<u>Pf.</u> $0v = (0+0)v = 0v + 0v$

$\Rightarrow 0v = 0v + 0v$

$\Rightarrow 0v = 0.$

$a0 = 0$ for any $a \in \mathbb{R}$.

Pf: $a0 = a(0+0) = a0 + a0 \Rightarrow a0 = 0$.

$(-1)v = -v \ \forall v \in V$.

<u>Pf:</u> $v + (-1)v = 1v + (-1)v = (1+(-1))v = 0v = 0$.

So $(-1)v$ is the additive inverse of v.

<u>Def:</u> A subset $U \subseteq V$ is a subspace of V if it is also a vector space with the same addition and scaling operations and the same additive identity.

<u>Prop:</u> A subset U is a subspace of V if and only if

1) $0 \in U$
2) $u, w \in U \Rightarrow u+w \in U$.
3) $a \in \mathbb{R}$ and $u \in U \Rightarrow au \in U$.

<u>Pf:</u> Certainly a subspace will satisfy these conditions. Now suppose a subset $U \subseteq V$ satisfies the above conditions. Associativity, commutativity, additive identity, multiplicative ident, distributive properties are all automatically satisfied.

For any $u \in U$, $(-1)u \in U$ by (3). But $(-1)u = -u$, so additive inverse property holds. Condition (2) ensures addition is well-defined. Similarly (3) ensures scalar mult. is well-defined. □

Span and Linear Independence

Def: A linear combination of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m$$

where each $a_i \in \mathbb{R}$.

Ex: Let $V = \mathbb{R}^3$, $v_1 = (2, 1, -3)$, $v_2 = (1, -2, 4)$.

$(17, -4, 2) = 6v_1 + 5v_2$, so $(17, -4, 2)$ is a linear combination of v_1 and v_2.

$(17, -4, 5)$ is not. Why?

$$\begin{cases} 17 = 2a_1 + a_2 \\ -4 = a_1 - 2a_2 \\ 5 = -3a_1 + 4a_2 \end{cases}$$ has no solutions.

The set of all linear combinations of $v_1, \ldots, v_m$ is called the span of $v_1, \ldots, v_m$:

$$\text{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbb{R}\}$$

Ex: in previous example

$(17, -4, 2) \in \text{span}(v_1, v_2)$

$(17, -4, 5) \notin \text{span}(v_1, v_2)$

Prop: The span of a list of vectors $v_1, \dots, v_m$ in V is the smallest subspace containing all vectors in the list.

("smallest" means that any other subspace containing $v_1, \dots, v_m$ must also contain $\text{span}(v_1, \dots, v_m)$)

Pf: $0 \in \text{span}(v_1, \dots, v_m)$. Why?

$\text{span}(v_1, \dots, v_m)$ is closed under addition. Why?

Also closed under scalar multiplication. Why?

Thus the span is a subspace.

Any other subspace containing $v_1, \dots, v_m$ must also contain $\text{span}(v_1, \dots, v_m)$ (why?), so $\text{span}(v_1, \dots, v_m)$ is indeed the smallest subspace containing each $v_1, \dots, v_m$. ☐

Def: $v_1, \dots, v_m$ spans V if

$$V = \text{span}(v_1, \dots, v_m).$$

Q: what is a spanning set/list for $\mathbb{R}^4$?

Def: A vector space is <u>finite dimensional</u> if it is the span of a (finite) list of vectors.

<u>Linear independence</u>: $v_1, \dots, v_m \in V$. $v \in \operatorname{span}(V)$.

By definition, there exist $a_1, \dots, a_m \in \mathbb{R}$ s.t.

$$v = a_1 v_1 + \dots + a_m v_m .$$

One natural question is whether this choice is unique. If $c_1, \dots, c_m \in \mathbb{R}$ are scalars that also satisfy

$$v = c_1 v_1 + \dots + c_m v_m ,$$ then

$$0 = (a_1 - c_1) v_1 + \dots + (a_m - c_m) v_m . \quad (1)$$

This exhibits 0 as a lin. comb. of $v_1, \dots, v_m$. Centainly,

$$0 = 0 v_1 + \dots + 0 v_m .$$

If this is the only way to write 0, then eqn (1) implies $a_1 = c_1$, $a_2 = c_2$, $\dots$, $a_m = c_m$, meaning there is a unique way to write v.

Def: A list $v_1, \dots, v_m$ in V is called linearly independent if $a_1 v_1 + \dots + a_m v_m = 0$ implies $a_1 = a_2 = \dots = a_m = 0$.

(i.e. $a_1 = \dots = a_m = 0$ is the <u>only</u> way to make $a_1 v_1 + \dots + a_m v_m = 0$)

Ex: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ are linearly indp.

why?

Def: $v_1, \dots, v_m$ are linearly dependent if they are not linearly independent.

Ex: $(2,3,1)$, $(1,-1,2)$, $(7,3,8)$ are linearly dependent since

$$2\begin{pmatrix}2\\3\\1\end{pmatrix} + 3\begin{pmatrix}1\\-1\\2\end{pmatrix} + (-1)\begin{pmatrix}7\\3\\8\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Lemma: Suppose $v_1, \dots, v_m$ is a linearly dependent list in V. Then for some $k \in \{1, 2, \dots, m\}$,

$$v_k \in \text{span}(v_1, \dots, v_{k-1}).$$

moreover, if v_k is removed from the list, the span of the remaining list equals $\text{span}(v_1, \dots, v_m)$

Pf: We have some $a_1, \dots, a_m$ not all zero s.t.

$$a_1 v_1 + \dots + a_m v_m = 0.$$

k be the largest index s.t. $a_k \neq 0$. Then

$$v_k = \frac{-a_1}{a_k} v_1 - \dots - \frac{a_{k-1}}{a_k} v_{k-1}, \quad (1)$$

so $v_k \in \text{span}(v_1, \dots, v_{k-1})$.

Thus if $v = c_1 v_1 + \dots + c_m v_m$, we can replace v_k with RHS of (1), so that removing v_k from $v_1, \dots, v_m$ doesn't change the span. $\square$

For technical reasons, we declare the span of an empty list to be $\{0\}$. This is to handle the case where $k=1$ in the above lemma.

<u>Proposition</u>: In a f.d.v.s, the length of every lin. indp list is $\leq$ the length of every spanning list.

<u>Pf:</u> Let $u_1, \dots, u_m$ be linearly indp. Let $w_1, \dots, w_n$ be a spanning list. (goal: $m \leq n$).

The idea is to swap out w's for u's one at a time, and to show that we can do this as long as we still have u's left, allowing us to conclude there are at least as many w's as there are u's.

<u>step 1:</u> Let $B = \{w_1, \dots, w_n\}$.

Since u_1 can be written as a linear combination of the w's by assumption,

$$u_1, w_1, \dots, w_n$$

is linearly dependent. $u_1 \neq 0$ by linear independence, which ensures we can remove one of the w's without changing the span. So we're removed a w and added a u.

Now insert u_2 after u_1 in this new list:

$$u_1, u_2, \text{remaining w's.}$$

Once again, this list is lin. dependent, and one of the vectors is a lin. combination of the ones before it. It can't be u_2 or u_1, so it must be one of the w's.

Discard this vector ... ☐

Ex: no $\overset{\text{lin indp}}{\vee}$ list of length 4 is lin. indp in $\mathbb{R}^3$.

E_1, E_2, E_3 is a spanning list. Thus any lin indp list must have length ≤ 3.

Ex: No list of length $3 \overset{\text{or less}}{\vee}$ spans $\mathbb{R}^4$.

E_1, E_2, E_3, E_4 is lin indp, so any spanning list needs at least 4 vectors.

Prop: Every subspace of a f.d.v.s. is f.d.

Pf: see Axler, ideas are similar. □

Bases

Def: A basis of V is a linearly indp set that spans V.

Ex: $e_1 = (1, 0, \ldots, 0)$
$e_2 = (0, 1, \ldots, 0)$ is a basis for $\mathbb{R}^n$.
$\vdots$
$e_n = (0, 0, \ldots, 1)$

b) $(1,2), (3,5)$ is a basis for $\mathbb{R}^2$.

c) $(1,2,-4), (7,-5,6)$ is linearly indp. in $\mathbb{R}^3$, but not a basis. Why?

d) $(1,2), (3,5), (4,13)$. spans $\mathbb{R}^2$, not a basis.

e) The list $(1,1,0), (0,0,1)$ is a basis for

$$\underbrace{\{(x,x,y) : x,y\in\mathbb{R}\}}_{\text{why is this a vector space?}}.$$

at least two ways to see this

f) $(1,-1,0), (1,0,-1)$ is a basis of

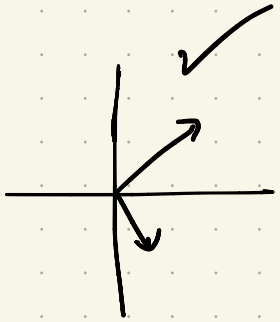
$$\{(x,y,z)\in\mathbb{R}^3 : x+y+z=0\}$$

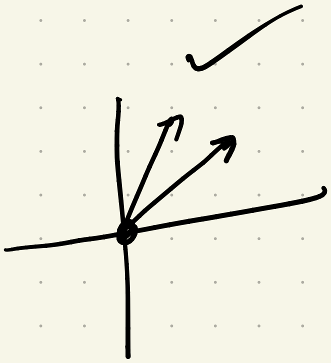
g) $1, x, x^2, \ldots, x^m$ is a basis of $\mathcal{P}^m(\mathbb{R})$.

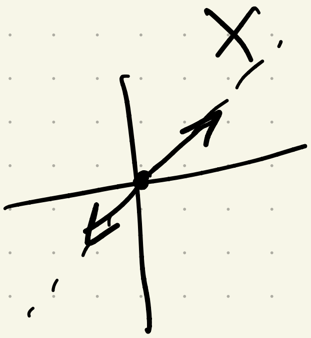
<u>Ex:</u> In $\mathbb{R}^2$, there are many possible bases

$(7,5), (-4,9)$ or $(1,2), (3,5)$.

Really, any two vectors that are not collinear.







<u>Prop:</u> A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ has exactly one representation of the form

$$v = a_1 v_1 + \cdots + a_n v_n, \quad a_i \in \mathbb{R}.$$

<u>Pf.:</u> Suppose $v_1, \ldots, v_n$ is a basis. Let $v \in V$. Then for some $a_i \in \mathbb{R}$

$$v = a_1 v_1 + \cdots + a_n v_n.$$

If we also have

$$v = c_1 v_1 + \cdots + c_n v_n,$$

$$0 = (a_1 - c_1) v_1 + \cdots + (a_n - c_n) v_n$$

$$\Rightarrow a_i = c_i \ \forall i \quad \text{since } v_1, \ldots, v_n \text{ lin indp.}$$

So every vector has exactly one representation as a lin comb. of $v_1, \ldots, v_n$.

Now suppose every $v \in V$ has a unique rep. wrt $v_1, \ldots, v_n$. Then $v_1, \ldots, v_n$ of course spans V. Also,

$$0 = 0 v_1 + \cdots + 0 v_n,$$

and by assumption, this is the only rep. Hence $v_1, \ldots, v_n$ are lin indp, so we have a basis. □

Ex: In $\mathbb{R}^2$, $(1,2), \boxed{(3,6)}, (4,7), \boxed{(5,9)}$ is a spanning list, but not a basis. If we remove vectors 2 and 4, though, we obtain a basis.

Prop: Every spanning list can be reduced to a basis.

Pf: "mark and sweep"

Start with spanning list $v_1, \dots, v_n$.

Step 1: If $v_1 = 0$, mark it for removal.

⋮

Step k: If $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$, mark v_k for deletion.

$$v_1 \quad \boxed{v_2} \quad \boxed{v_3} \quad v_4 \quad \dots \quad v_{n-1} \quad \boxed{v_n}$$

now delete the marked vectors.

The resulting list still spans V.

Also, no vector is in the span of the previous ones, so the list is lin indp. □

Cor: Every fdvs has a basis.

Prop: Every linearly independent set in a fdvs can be extended to a basis.

Proof: Let $u_1, \dots, u_m$ be lin indp.
By assumption, V has some spanning list $w_1, \dots, w_n$. Apply the reduction process to

$$u_1, \dots, u_m, w_1, \dots, w_n.$$

No u_i's will be deleted since no u_i is in the span of the ones before it. □

Ex: $\mathbb{R}^3$ $\quad (2,3,4), (9,6,8)$

$$(1,0,0), (0,1,0), (0,0,1)$$

$$(2,3,4), (9,6,8), (1,0,0), (0,1,0), (0,0,1)$$

Sums of subspaces

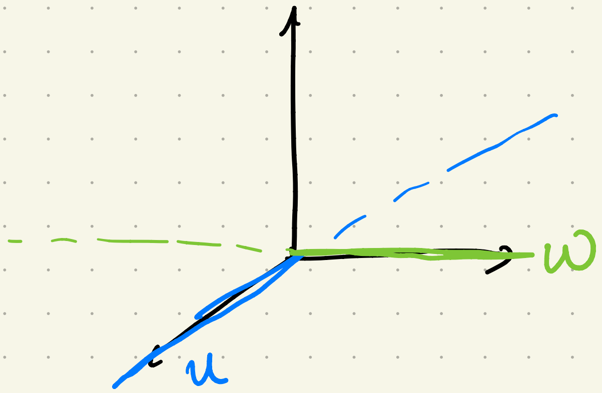
Let $V_1, \dots, V_m$ be subspaces of V. The <u>sum</u> of $V_1, \dots, V_m$ is

$$V_1 + \dots + V_m = \{v_1 + \dots + v_m : v_1 \in V_1, \dots, v_m \in V_m\}$$

Ex: $\mathbb{R}^3$

$U = \{(x,0,0) : x \in \mathbb{R}\}$, $W = \{(0,y,0) : y \in \mathbb{R}\}$

Then $U + W = \{(x,y,0) : x, y \in \mathbb{R}\}$



Prop: $V_1, \dots, V_m$ subspaces of V.

$V_1 + \dots + V_m$ is the smallest subspace containing each of $V_1, \dots, V_m$.

Def: $V_1 + \dots + V_m$ is a <u>direct sum</u> if each $v \in V_1 + \dots + V_m$ can be written in only one way as $v = v_1 + \dots + v_m$, $v_k \in V_k$.

In this case, we write

$$V_1 \oplus \dots \oplus V_m.$$

Ex: $U = \{(x,y,0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$

$W = \{(0,0,z) \in \mathbb{R}^3 : z \in \mathbb{R}\}$

Then $\mathbb{R}^3 = U \oplus W$.

Pf: Certainly $\mathbb{R}^3 = U + W$.

Now if $u_1 + w_1 = u_2 + w_2$

$(x_1, y_1, z_1) = (x_2, y_2, z_2)$

$\Rightarrow x_1 = x_2, y_1 = y_2, z_1 = z_2$

$\Rightarrow u_1 = u_2$ and $w_1 = w_2$. □

$u_1, u_2 \in U$

$w_1, w_2 \in W$.

$u_1 = (x_1, y_1, 0)$

$u_2 = (x_2, y_2, 0)$

$w_1 = (0, 0, z_1)$

$w_2 = (0, 0, z_2)$

Ex: a sum that is not direct

$V_1 = \{(x,y,0) : x, y \in \mathbb{R}\}$

$V_2 = \{(0,0,z) : z \in \mathbb{R}\}$

$V_3 = \{(0,y,y) : y \in \mathbb{R}\}$

Again, $\mathbb{R}^3 = V_1 + V_2 + V_3$,

but $(0,0,0) = (0,1,0) + (0,0,1) + (0,-1,-1)$

$(0,0,0) = (0,0,0) + (0,0,0) + (0,0,0)$.

Prop: $V = V_1 \oplus \cdots \oplus V_m$ if and only if the only way to write 0 is by taking 0 for each v_k in

$$0 = v_1 + v_2 + \cdots + v_m.$$

Pf: . . . $\square$

Prop: U, W subspaces of V. Then

$$U + W \text{ direct} \iff U \cap W = \{0\}.$$

Pf: ($\Rightarrow$) $v \in U \cap W$. $0 = v + (-v) \in U + W$.

But $U \oplus W$, so $v = -v = 0$. So $U \cap W = \{0\}$.

($\Leftarrow$) Suppose $U \cap W = \{0\}$.

$$0 = u + w, \quad u \in U, \; w \in W.$$

$$\underset{\overset{\cap}{U}}{-u} = w \implies w \in U.$$

So $w \in U \cap W = \{0\}$. So $w = 0$, and in turn $u = 0$. $\square$

Prop: V = fdvs, U subspace. Then there is a subspace W of V s.t.

$$V = U \oplus W.$$

Idea: extend basis $u_1, \dots, u_m$ of U to basis $u_1, \dots, u_m, w_1, \dots, w_n$ of V.

Set $W = \text{span}(w_1, \dots, w_n)$. $\square$

Dimension

Prop: Any two bases of a fdvs have the same length.

Pf: Let B_1 and B_2 be bases.

$$\underset{\substack{\text{lin}\\\text{indp}\\\text{list}}}{\text{length}(B_1)} \le \underset{\substack{\text{spanning}\\\text{list}}}{\text{length}(B_2)}$$

$$\underset{\substack{\text{lin}\\\text{indp.}\\\text{list}}}{\text{length}(B_2)} \le \underset{\substack{\text{spanning}\\\text{list}}}{\text{length}(B_1)} \qquad \square$$

Ex: $\dim \mathbb{R}^n = n$.

$\dim \mathcal{P}_m(\mathbb{R}) = m+1$.

Prop: if U is a subspace of V = fdvs, then $\dim U \leq \dim V$.

Pf: ... $\square$

Prop: V = fdvs. Every lin indp list of length $\dim V$ is a basis of V.

Pf: $\dim V = n$. Let $v_1, \dots, v_n$ be lin indp. $v_1, \dots, v_n$ can be extended to a basis. But any basis has length n, so that the extension is to add nothing. So $v_1, \dots, v_n$ is a basis. $\square$

Prop: V = fdvs, U subspace. If $\dim U = \dim V$, then $U = V$.

Pf: Let $n = \dim U = \dim V$. Let $u_1, \dots, u_n$ be a basis for U. From prev. prop., $u_1, \dots, u_n$ is also a basis for V, so it spans V, meaning $U = V$.

Ex: $(5,7),(4,3) \in \mathbb{R}^2$. This is a lin. indp list since neither is a scalar mult. of the other. $\dim \mathbb{R}^2 = 2$, so we know this list is a basis.

We do not need to bother checking the span is $\mathbb{R}^2$.

Ex: Let $V = \mathcal{P}_3(\mathbb{R})$.

$$U = \{ p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0 \}.$$

Find a basis for U.

Note $1, (x-5)^2$, and $(x-5)^3$ lie in U.

Suppose $a + b(x-5)^2 + c(x-5)^3 = 0 \quad \forall x \in \mathbb{R}$.

We see cx^3 is the cubic term on LHS.

$\Rightarrow c = 0$.

This in turn implies the x^2 term is bx^2.

$\Rightarrow b = 0$.

$\Rightarrow a = 0$.

side note:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Why?

So $\dim U \geq 3$.

So $\quad 3 \leq \dim U \leq \dim \mathcal{P}_3(\mathbb{R}) = 4.$

$x \notin U$. So U is not all of $\mathcal{P}_3(\mathbb{R})$, meaning $\dim U \neq 4. \Rightarrow \dim U = 3.$

Conclude $1, (x-5)^2, (x-5)^3$ is a basis of U.

$$\left(\begin{aligned}
p &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
p'(x) &= a_1 + 2a_2 x + 3a_3 x^2 \\
p'(5) &= a_1 + 10a_2 + 75a_3 = 0 \\
& a_0 + (-10a_2 - 75a_3)x + a_2 x^2 + a_3 x^3 \\
&= a_0 + a_2(-10x + x^2) + a_3(-75x + x^3)
\end{aligned}\right)$$

Prop: V: fdvs. If a spanning list has length $\dim V$, then it is a basis of V.

Proof: A spanning list can be reduced to a basis. But the remaining list must have length $\dim V$. $\square$

Prop.: $\dim(V_1 + V_2) = \dim V_1 + \dim V_2$
$- \dim(V_1 \cap V_2)$.