Green's Theorem

$$F(x,y) = (p(x,y),\ q(x,y)) \quad \left(\begin{array}{c}\text{assume} \\ \text{smooth functions}\end{array}\right)$$

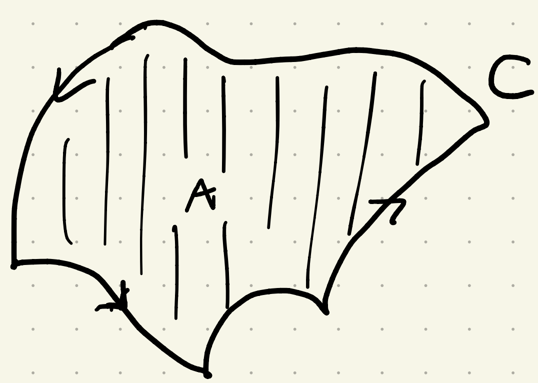
$$\int_C F = \int_a^b F(c(t)) \cdot c'(t)\,dt = \int_C p(x,y)\,dx + q(x,y)\,dy$$

abbr. as $\int_C F = \int_C p\,dx + q\,dy$

reasonable notation since $F(c(t)) \cdot c'(t)\,dt$
$= p\frac{dx}{dt} + q\frac{dy}{dt}$.

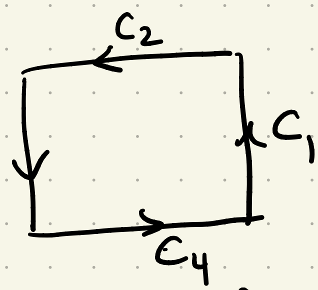
Thm 1.1. (Green's Theorem) Let (p, q) be a vector field on A. Let A be the interior of a closed path C oriented CCW. Then

$$\int_C p\,dx + q\,dy = \iint_A \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dy\,dx.$$



Proving it in full generality is difficult, but we can show it for certain types of regions.

Suppose A is a rectangular region $[a,b]\times[c,d]$.



(1) $$\iint_A \frac{\partial p}{\partial y}\,dy\,dx = \int_a^b \int_c^d \frac{\partial}{\partial y} p(x,y)\,dy\,dx$$

$$= \int_a^b p(x,y)\Big|_{y=c}^{y=d} dx = \int_a^b \big(p(x,d) - p(x,c)\big)\,dx$$

$$\mathcal{C} = (C_1, C_2, C_3, C_4)$$

$$= -\int_{C_2} p\,dx - \int_{C_4} p\,dx$$

on vertical segments, $dx = 0 \Rightarrow \int_{C_1} p\,dx = 0$

$\int_{C_3} p\,dx = 0$

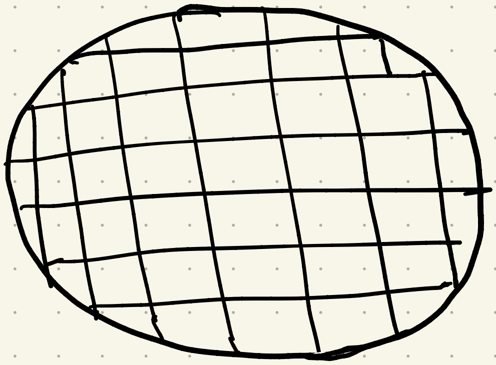
so $$\iint_A \frac{\partial p}{\partial y}\,dy\,dx = \int_{\mathcal{C}} p\,dx$$

(2) $$\iint_A \frac{\partial q}{\partial x}\,dx\,dy = \int_c^d \int_a^b \frac{\partial q}{\partial x}\,dx\,dy = \int_c^d \left[q(b,y) - q(a,y)\right] dy$$

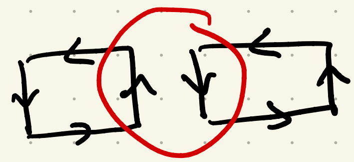
$$= \int_{C_1} q\,dy + \int_{C_3} q\,dy = \int_{\mathcal{C}} q\,dy.$$

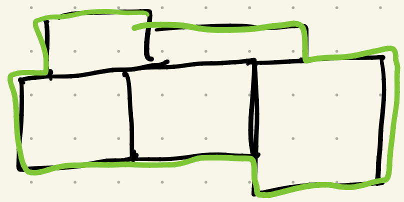
(1) + (2) $$\Longrightarrow \iint_A \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dy\,dx = \int_{\mathcal{C}} p\,dx + q\,dy. \ \square$$

We can then approximate regions by rectangles.



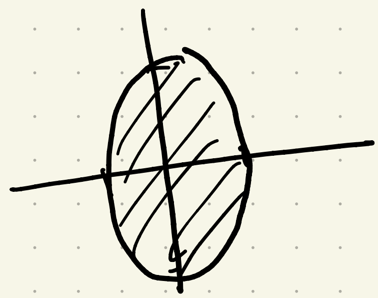
notice that on two adjacent rectangles, cancellations occur, and you're left with the boundary.





Ex 1: $F(x,y) = (\overset{p}{y+3x}, \overset{q}{2y-x})$

$C: 4x^2+y^2=4$

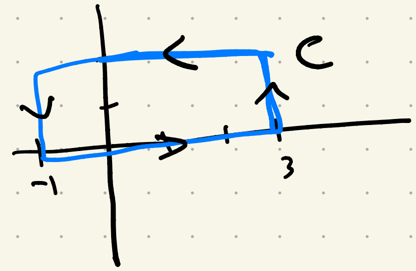


$\frac{\partial p}{\partial y} = 1, \ \frac{\partial q}{\partial x} = -1$

$$\int_C p\,dx + q\,dy = \iint_A -2\,dy\,dx = -2 \cdot \text{Area}(A)$$

$$= -2 \cdot \pi \cdot 1 \cdot 2 = -4\pi.$$

Ex 2: $F = (3xy, x^2)$



$\int p\,dx + q\,dy$

$$\int_C p\,dx + q\,dy = \iint_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dy\,dx$$

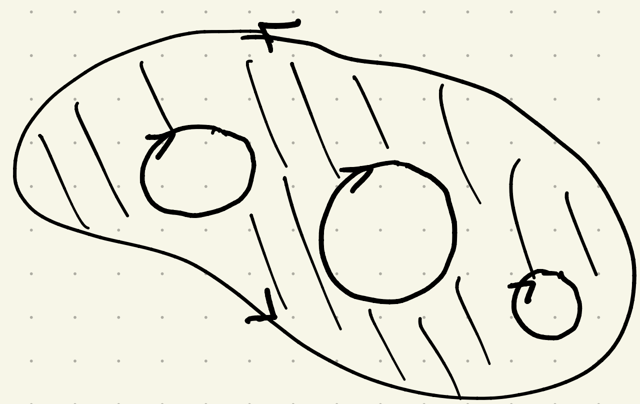
$$= \int_{-1}^{3}\int_{0}^{2} (2x - 3x)\, dy\,dx = \int_{-1}^{3}\int_{0}^{2} -x \, dy\,dx$$

$$= \int_{-1}^{3} -2x \, dx = -x^2\Big|_1^3 = -9 + 1 = -8.$$

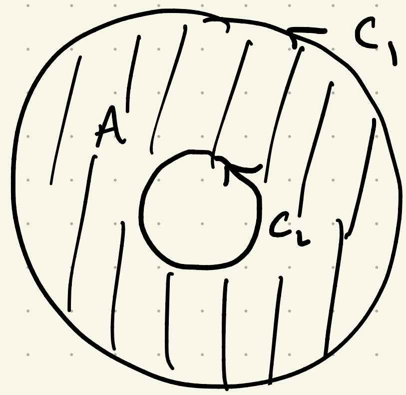
This is much better than parametrizing the boundary.

Green's Thm, general version: Let A be a region in the plane whose boundary consists of a finite # of curves, each one oriented s.t. A lies to the left.

Then
$$\int_C p\,dx + q\,dy = \iint_A \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dy\,dx$$



Ex: Let A be the region between two concentric circles with CCW orientation.



Applying Green's theorem here would require us to reverse C_2, so the integral in Green's formula would be

$$\int_{C_1} p\,dx + q\,dy + \int_{C_2^-} p\,dx + q\,dy = \iint_A \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dy\,dx$$

$$\parallel$$

$$\int_{C_1} p\,dx + q\,dy - \int_{C_2} p\,dx + q\,dy$$

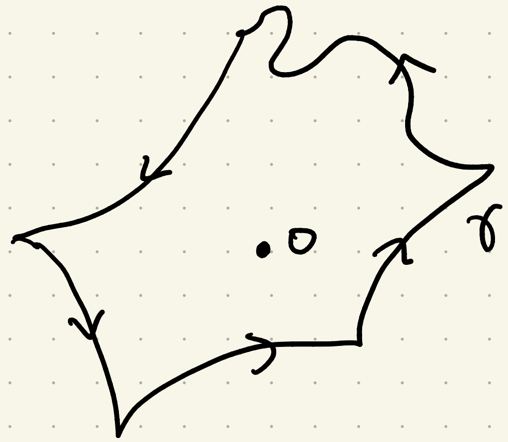
If $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$, then we have

$$\int_{C_1} p\,dx + q\,dy = \int_{C_2} p\,dx + q\,dy.$$

If F is conservative, we always have $D_2 p = D_1 q$.

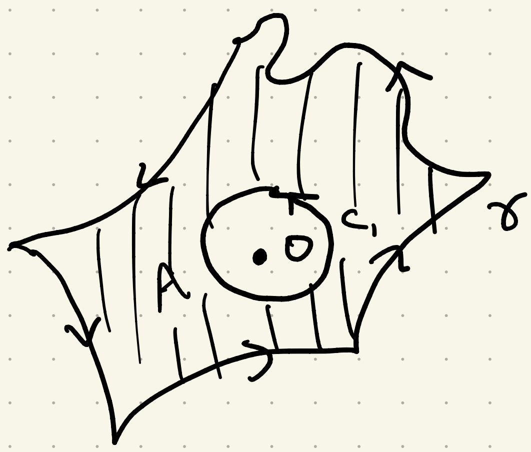
But we can have $D_2 p = D_1 q$ for a nonconservative vector field.

Ex: $G = \left(\frac{-y}{x^2+y^2} , \frac{x}{x^2+y^2} \right)$



$$\int_\gamma G = ?$$

Note: Green's Thm doesn't immediately apply since the interior of C contains a pt. for which G is undefined.



Introduce C_1, a circle centered at O.

Then $\displaystyle\int_\gamma G = \underbrace{\int_{C_1} G}_{\substack{\text{we know} \\ \text{this guy.} \\ = 2\pi}}$

Note: If γ does not enclose the origin, then

$\displaystyle\int_\gamma G = 0$. Why?

2) Divergence and Rotation

$F = (p, q)$ vector field $p, q : U \subseteq \mathbb{R}^2 \to \mathbb{R}$

The divergence of F is

$$\operatorname{div} F = D_1 p + D_2 q = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}$$

The rotation of F is

$$\operatorname{rot} F = D_1 q - D_2 p = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}.$$

Note that $\operatorname{rot} F$ is exactly the quantity appearing in Green's theorem.

It may help in the following discussion to imagine that F is describing a fluid flow in a region.

Green's Theorem states

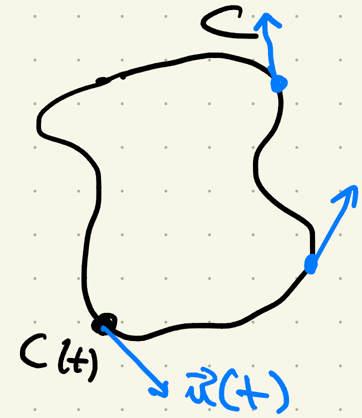
$$\iint_A \operatorname{rot} F \, dy\,dx = \int_a^b F(c(t)) \cdot c'(t)\, dt$$

where A is the region inside C oriented CCW.

Now, $\|C'(t)\| = \frac{ds}{dt}$, the speed.

Here $s(t)$ is the distance traveled.

Let $\vec{u}$ be a unit vector in the tangential direction of C at time t.



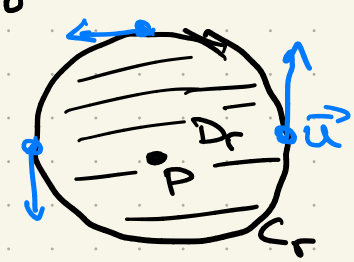
$$C'(t) = \vec{u}(t)\frac{ds}{dt}$$

Then

$$\iint_A \text{rot}\, F \, dy\, dx = \int_C F \cdot \vec{u} \, ds$$

↳ the more aligned F is with the velocity, the more this integral is picking up.

Thm 2.1 Let D_r be the disk of radius r at a point P. Let C_r be the boundary of D_r. Let F be a vector field defined on the closed disk. Let $A(r) = \pi r^2$.



Then

$$(\text{rot}\, F)(P) = \lim_{r \to 0} \frac{1}{A(r)} \int_{C_r} F \cdot \bar{u}$$

<u>Proof:</u> Fix $x = (x, y)$ in the disk.

$$\text{rot}\, F(x) = \text{rot}\, F(P) + h(x)$$

where $\lim_{x \to P} h(x) = 0$ (this can be done by continuity).

$$\frac{1}{A(r)} \int_{C_r} F \cdot \vec{u}\, ds = \frac{1}{A(r)} \iint_{D_r} \text{rot}\, F \, dy\, dx$$

$$= \frac{1}{A(r)} \iint_{D_r} \underbrace{\text{rot}\, F(P)}_{\text{constant}} \, dy\, dx + \frac{1}{A(r)} \iint_{D_r} h(x,y)\, dy\, dx$$

$$= \text{rot}\, F(P) + \frac{1}{A(r)} \iint_{D_r} h(x,y)\, dy\, dx$$

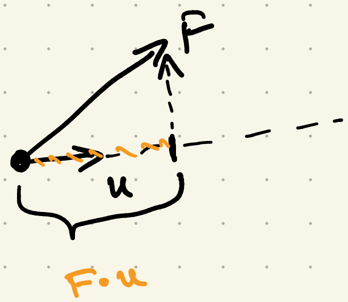
$$\left| \frac{1}{A(r)} \iint_{D_r} h(x,y)\, dy\, dx \right| \le \frac{1}{A(r)} \iint_{D_r} |h(x,y)|\, dy\, dx$$

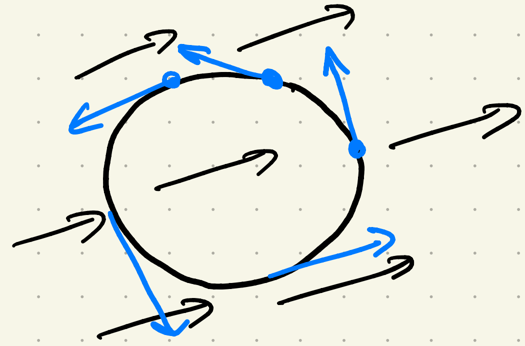
$$\le \frac{1}{A(r)} \iint_{D_r} \max_{D_r} |h(x,y)|\, dy\, dx$$

$$= \max_{D_r} |h(x,y)| \frac{1}{A(r)} \iint_{D_r} dy\, dx = \max_{D_r} |h(x,y)| \to 0$$

as $r \to 0$. $\square$

$F \cdot \vec{u}$ is the component of F in the tangential direction.

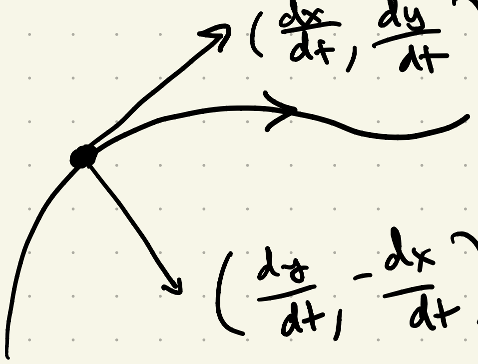




Thus $\text{rot}\, F(P)$ gives a quantitative measure of how much the vector field circulates around a given point.

$C(t) = (x(t), y(t)) \quad a \le t \le b$

$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = c'(t)$

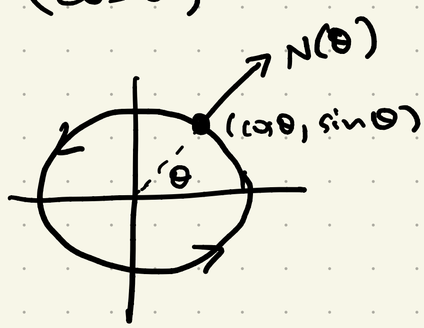


$\left(\frac{dy}{dt}, -\frac{dx}{dt}\right) = N \rightarrow$ <u>right normal vector.</u>

<u>Ex:</u> $c(\theta) = (\cos\theta, \sin\theta)$

$N(\theta) = (\cos\theta, \sin\theta)$

The position vector for a parametrized circle is always normal to the circle itself.



We can now look at $F(C(t)) \cdot N(t)$
rather than $F(C(t)) \cdot C'(t)$

Thm: A = region, interior of a closed CCW curve C. F vector field on A. Then

$$\iint_A (\operatorname{div} F)\, dy\, dx = \int_C F(C(t)) \cdot N(t)\, dt$$

Pf: your HW.

$C'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right)$, so

$$\|N(t)\| = \|C'(t)\| = \overset{\text{speed}}{v(t)}$$

dist traveled $s(t) = \int v(t)\, dt$, so $\frac{ds}{dt} = v(t)$

$$\vec{n}(t) = \frac{N(t)}{\|N(t)\|}.$$

$$N(t) = \|N(t)\|\, \vec{n}(t) = \frac{ds}{dt} \vec{n}(t)$$

so $$\iint_A \operatorname{div} F\, dy\, dx = \int_C F \cdot \vec{n}\, ds$$

<u>Thm:</u> $$(\operatorname{div} F)(P) = \lim_{r \to 0} \frac{1}{A(r)} \int_{C_r} F \cdot \vec{n} \, ds$$

<u>Pf:</u> Similar

$F \cdot \vec{n}$ is the component of f normal to the curve. Thus the integral on the right side above is measuring some kind of outward/inward flow.