Surface Parametrizations

A curve can be described by an algebraic equation such as

$$x^2 + y^2 = 1,$$

or it can be given parametrically

e.g. $(\cos t, \sin t)$ $0 \le t \le 2\pi$.

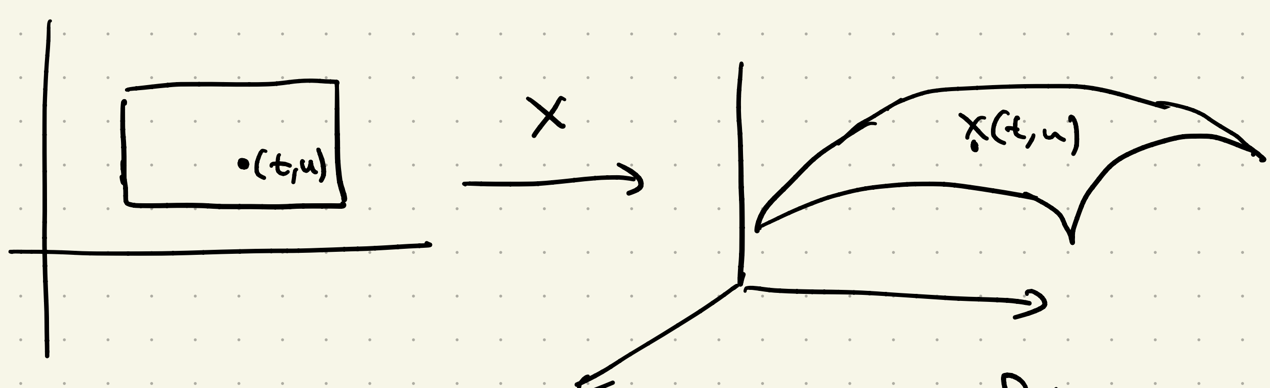
A parametrization $C(t) : [a,b] \longrightarrow \mathbb{R}^2$ can be thought of as a way of "sewing" the interval $[a,b]$ into the fabric that is $\mathbb{R}^2$.

We can move this idea one dimension up.

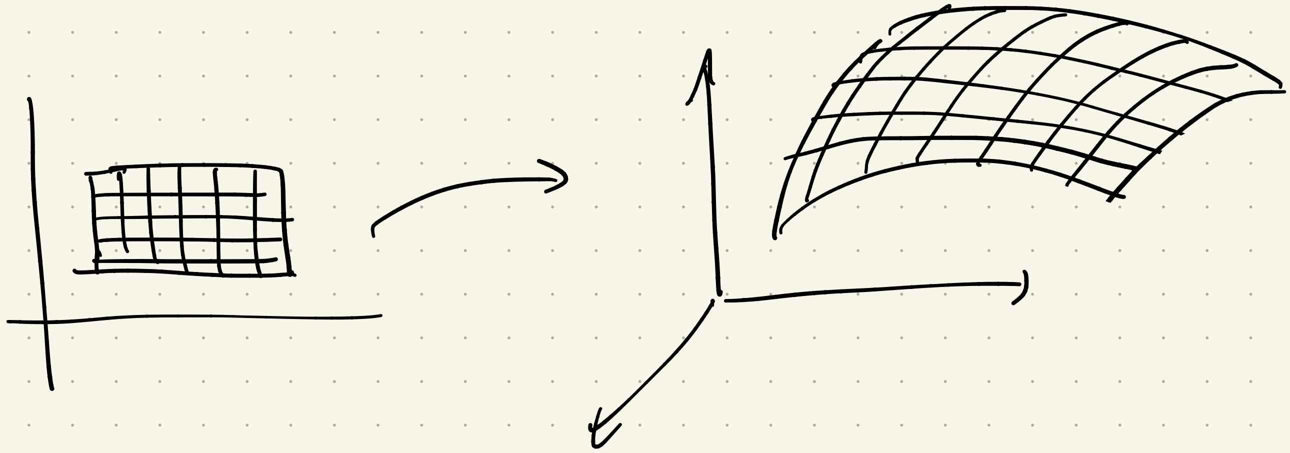
$$R \subseteq \mathbb{R}^2, \quad X : R \longrightarrow \mathbb{R}^3$$

$$X(t,u) = (x_1(t,u), x_2(t,u), x_3(t,u))$$

$$x_i : R \longrightarrow \mathbb{R}$$



Now we're sewing a "sheet" into the fabric of space

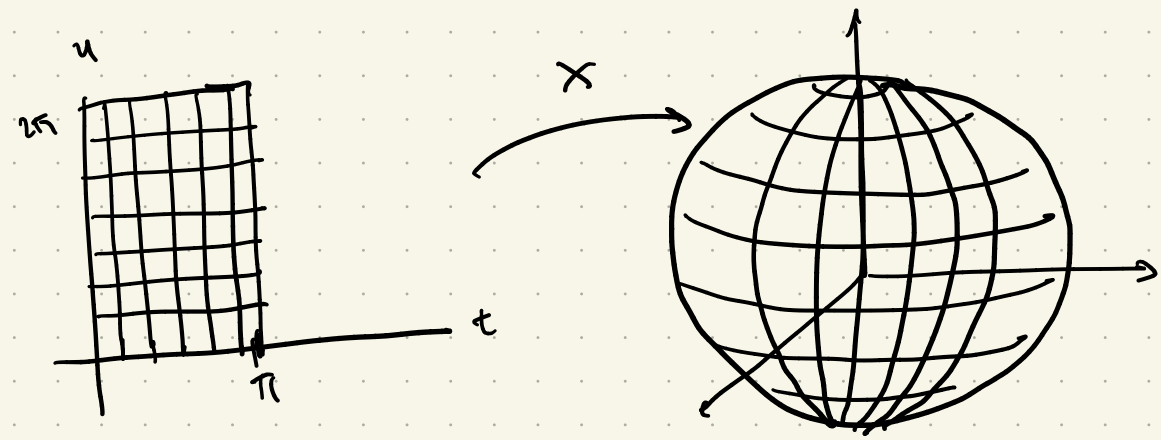


Ex: $X(t,u) = (\rho \sin t \cos u, \rho \sin t \sin u, \rho \cos t)$

$\rho \in \mathbb{R}$ fixed

$0 \le t \le \pi$

$0 \le u < 2\pi$.



Ex: torus

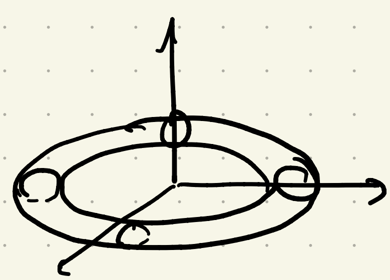
$x = (a + b\cos\varphi)\cos\theta$

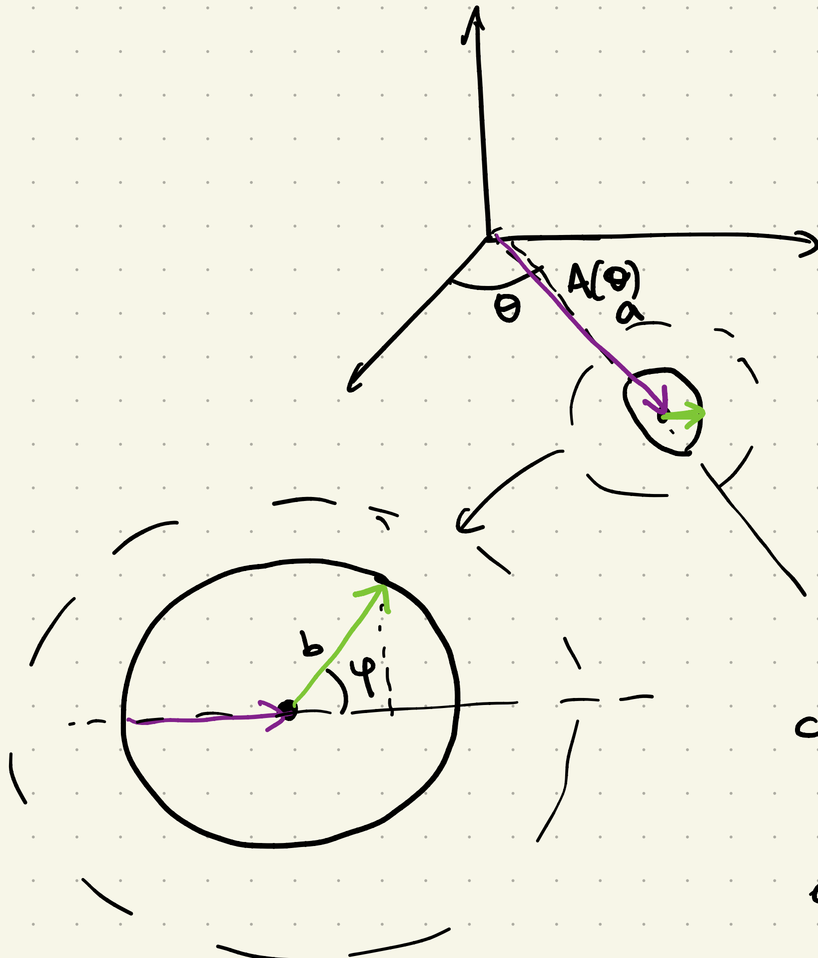
$y = (a + b\cos\varphi)\sin\theta$

$z = b\sin\varphi$.

$\varphi \in [0, 2\pi]$

$\theta \in [0, 2\pi]$





$A(\theta) =$
$(a\cos\theta, a\sin\theta, 0)$

$\hat{A}(\theta) = (\cos\theta, \sin\theta)$

Break green vector B into component along A(θ) and the vertical component.

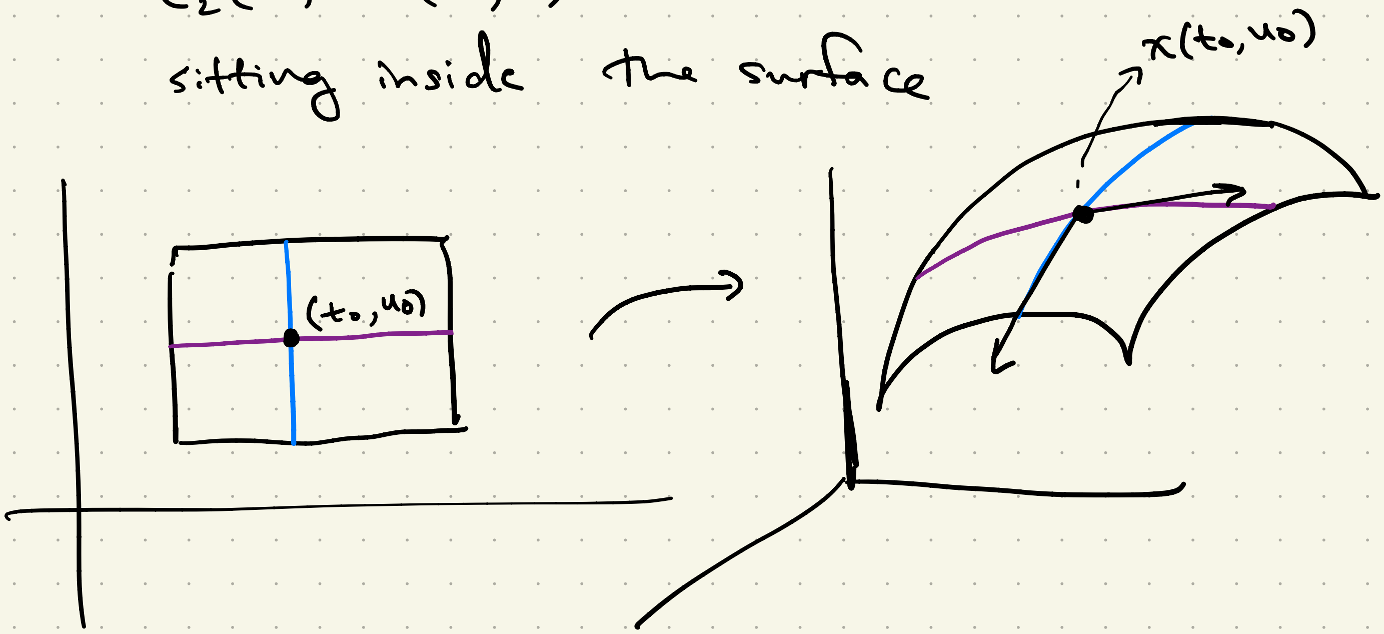
$$B_{A(\theta)} = b\cos\varphi\,(\cos\theta, \sin\theta, 0)$$

$$B_{vert} = b\sin\varphi\,(0, 0, 1)$$

$$A(\theta) + B_{A(\theta)} + B_{vert}$$
$$= (a\cos\theta + b\cos\varphi\cos\theta,\ a\sin\theta + b\cos\varphi\sin\theta,\ b\sin\varphi)$$

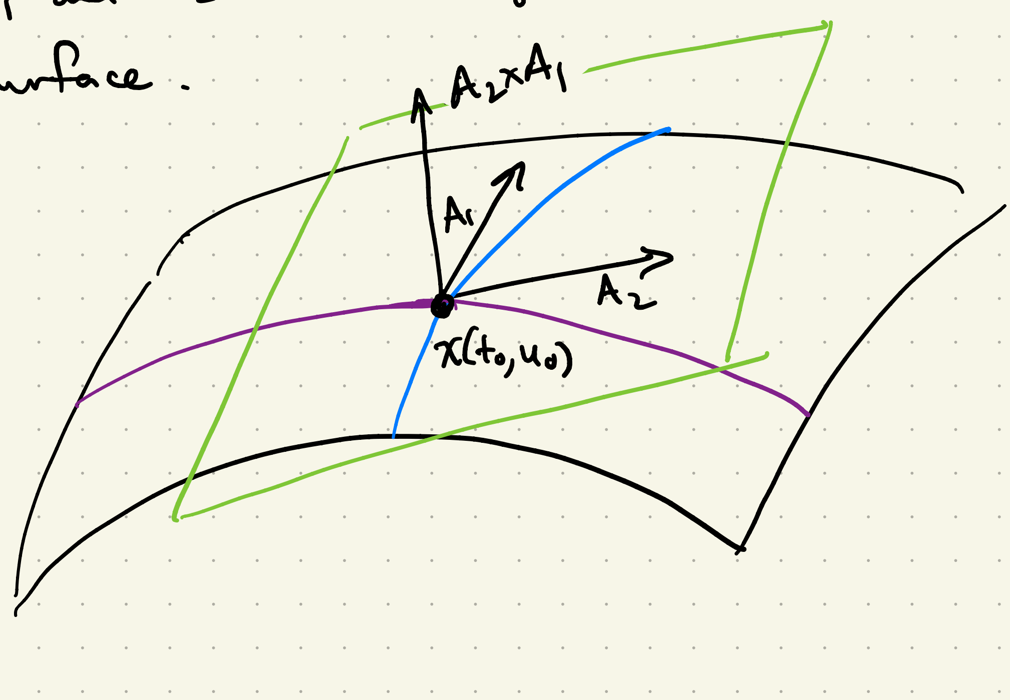
Let R be a region in $\mathbb{R}^2$ and $X(t,u)$ a parametrized surface.

The curves $C_1(t) = X(t,u)$ and $C_2(u) = X(t,u)$ are then curves sitting inside the surface



$$A_1 = \left.\frac{\partial X}{\partial t}\right|_{(t_0,u_0)}, \quad A_2 = \left.\frac{\partial X}{\partial u}\right|_{(t_0,u_0)}$$

A_1 and A_2 are tangent vectors to the surface.

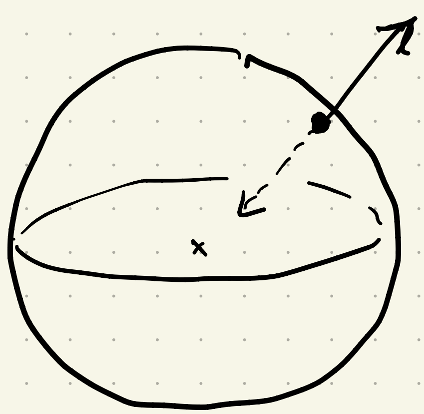


The tangent plane of the surface at this point $x(t_0, u_0)$ is the plane through $x(t_0, u_0)$ that is parallel to both A_1 and A_2.

Equivalently, it is the plane through $x(t_0, u_0)$ normal to $A_1 \times A_2$.

$$\boxed{N(t,u) = \frac{\partial x}{\partial t} \times \frac{\partial x}{\partial u}} \longrightarrow \text{normal to the surface at each } (t,u)$$

The order in which we take the cross product flips the normal vector. For a closed orientable surface, one of these points inward, the other outward.



Möbius band

$$\vec{n} = \frac{N}{\|N\|} = \frac{\frac{\partial x}{\partial t} \times \frac{\partial x}{\partial u}}{\left\| \frac{\partial x}{\partial t} \times \frac{\partial x}{\partial u} \right\|}$$

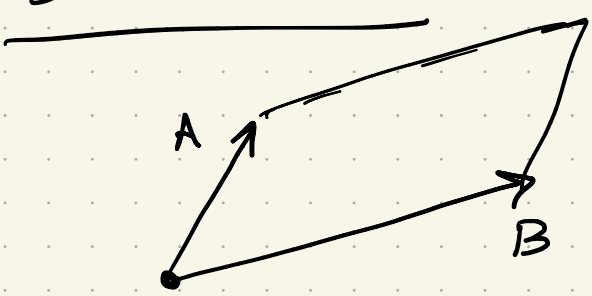
Ex: $X(\varphi, \theta) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi)$

$$\left(N(\varphi, \theta) = \rho \sin\varphi \, X(\varphi, \theta) \right)$$

$$\left(\|N(\varphi, \theta)\| = \rho^2 \sin\varphi \right)$$

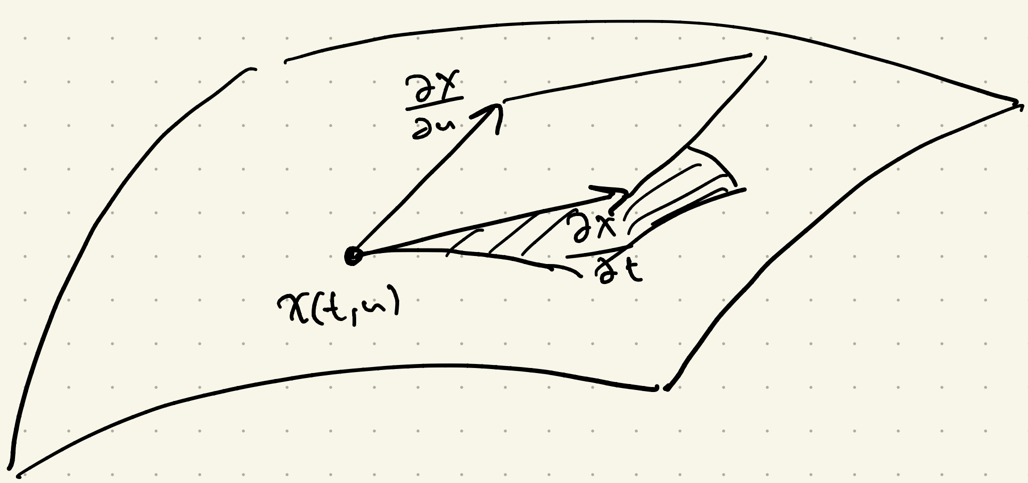
$$\vec{n} = \frac{1}{\rho} X(\varphi, \theta)$$

Surface Area

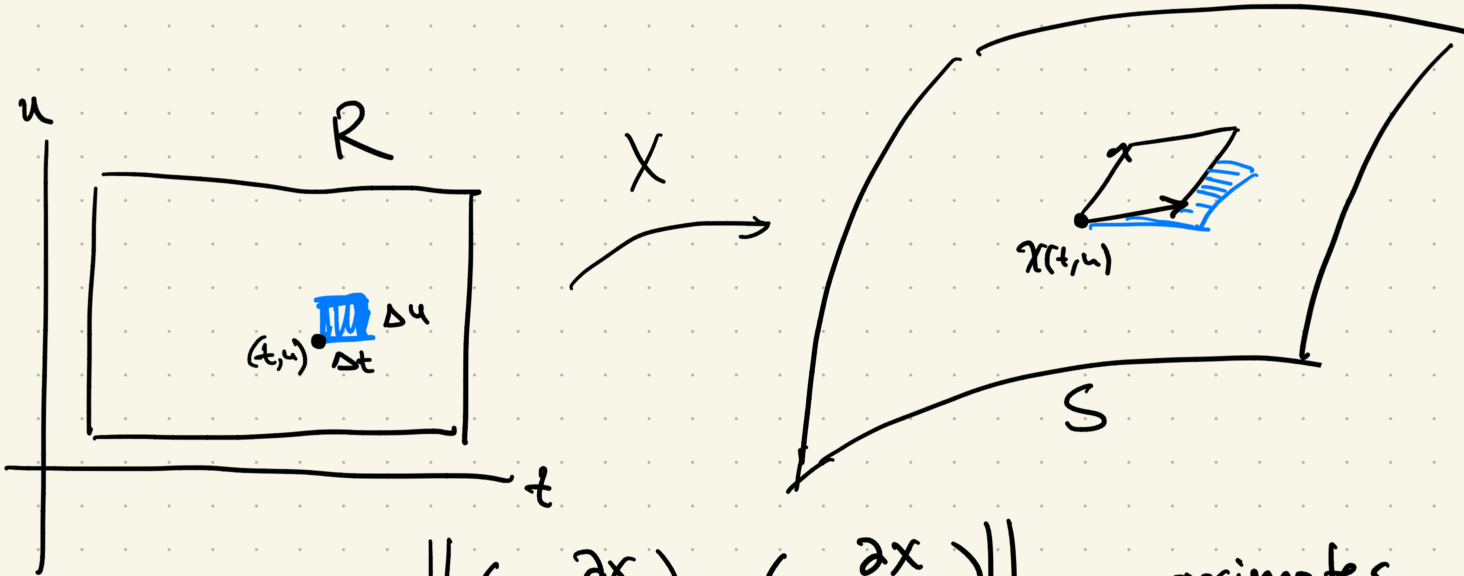


area of parallelogram

$\|A \times B\| = \|A\| \, \|B\| \sin\theta$



For small changes $\Delta t, \Delta u$ in t, u

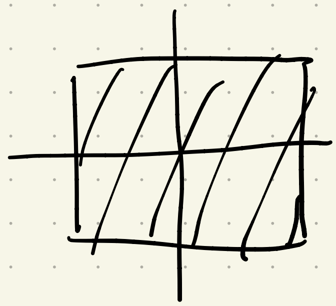


the area $\left\| \left(\Delta u \frac{\partial x}{\partial u}\right) \times \left(\Delta t \frac{\partial x}{\partial t}\right) \right\|$ approximates the blue patch quite well.

$$\boxed{\text{Area}(S) = \iint_S d\sigma = \iint_R \left\| \frac{\partial x}{\partial t} \times \frac{\partial x}{\partial u} \right\| dt\, du}$$

Random Aside: computing $\int_{-\infty}^{\infty} e^{-x^2} dx$

$$\int_{-R}^{R} \left[\int_{-R}^{R} e^{-x^2-y^2} dy \right] dx \quad 1$$



$$= \int_{-R}^{R} e^{-x^2} \left[\int_{-R}^{R} e^{-y^2} dy \right] dx$$

$$= \left(\int_{-R}^{R} e^{-x^2} dx \right) \left(\int_{-R}^{R} e^{-y^2} dy \right)$$

$$= \left(\int_{-R}^{R} e^{-x^2} dx \right)^2$$

From HW, $1 \to \pi$ as $R \to \infty$.

So $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Ex: $\chi(\varphi,\theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$

$$\frac{\partial x}{\partial \varphi} = (\rho\cos\varphi\cos\theta, \rho\cos\varphi\sin\theta, -\rho\sin\varphi)$$

$$\frac{\partial x}{\partial \theta} = (-\rho\cos\varphi\sin\theta, \rho\cos\varphi\cos\theta, 0)$$

$$\frac{\partial x}{\partial \varphi} \times \frac{\partial x}{\partial \theta} = \begin{vmatrix} E_1 & E_2 & E_3 \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \\ -\rho\cos\varphi\sin\theta & \rho\cos\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \dots = \rho^2\sin\varphi$$

$$\text{Area} = \int_0^{2\pi}\int_0^{\pi} \rho^2 \sin\varphi \, d\varphi \, d\theta$$

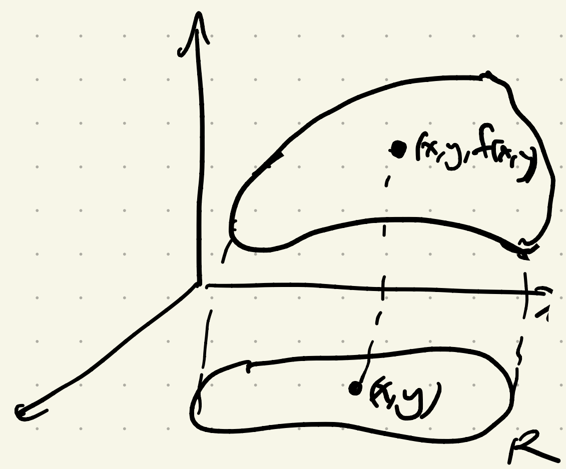
$$= \rho^2 \int_0^{2\pi} -\cos\varphi \Big|_{\varphi=0}^{\varphi=\pi} d\theta = \rho^2 \cdot 2 \int_0^{2\pi} d\theta = 4\pi\rho^2.$$

A noteworthy special case:

Let $z = f(x,y)$ be a surface.

Then we have the natural parametrization

$$X(x,y) = (x, y, f(x,y))$$



$$\frac{\partial X}{\partial x} = \left(1, 0, \frac{\partial f}{\partial x}\right)$$

$$\frac{\partial X}{\partial y} = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

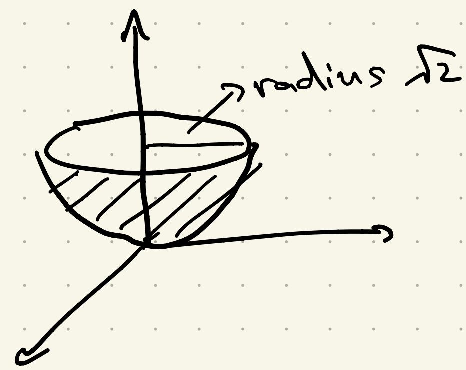
$$\frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y} = \left(-\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, 1\right)$$

$$\left\| \frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y} \right\| = \sqrt{1 + f_x^2 + f_y^2}$$

$$\text{Surface area} = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

Ex.: Find the area of the paraboloid

$z = x^2 + y^2$ where $0 \le z \le 2$.



$$\text{Area} = \iint_R \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1+4r^2}\, r\, dr\, d\theta$$

$$= \dots = 13\pi/3.$$

Integral of a function over a surface

$$d\sigma = \left\| \frac{\partial X}{\partial t} \times \frac{\partial X}{\partial u} \right\| dt\, du$$

differential surface area

$$\iint_S \psi \, d\sigma = \iint_R \psi(X(t,u)) \left\| \frac{\partial X}{\partial t} \times \frac{\partial X}{\partial u} \right\| dt\, du$$

Ex: Let S be the surface defined by

$z = x^2 + y$, $\quad 0 \le x \le 1$, $-1 \le y \le 1$

Find $\iint_S x \, d\sigma$

$$X(x,y) = (x, y, x^2+y)$$

$$d\sigma = \sqrt{1 + (2x)^2 + 1} = \sqrt{2+4x^2}\, dA$$

$$\int_{-1}^{1}\int_{0}^{1} x\sqrt{2+4x^2}\, dx\, dy$$

$$= \int_{-1}^{1}\int_{0}^{1} x(2+4x^2)^{1/2}\, dx\, dy$$

$$= 2\int_0^1 x(2+4x^2)^{1/2}\, dx$$

$$= 2\cdot\frac{2}{3}(2+4x^2)^{3/2}\cdot\frac{1}{8}\Big|_0^1$$

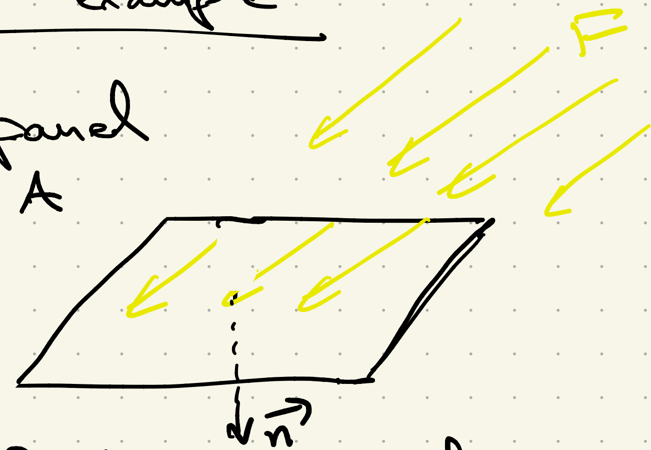
$$= \frac{1}{6}(2+4x^2)^{3/2}\Big|_0^1 = \frac{1}{6}\left(6^{3/2} - 2^{3/2}\right)$$

One can think of ψ as the density (kg/m^2) at a given point of the surface.

Integral of a vector field over a surface.

<u>simple example</u>

Solar panel w/ area A



solar energy density on earth is $\approx 1370 \ \mathrm{W/m^2}$

For a fixed time of day and a small region on earth, the field associated with the solar field is constant F.

The energy the panel is absorbing (assuming it's absorbing 100%) is then

$$(F \cdot \vec{n})A = F \cdot (A\vec{n}) \quad \text{(watts)}$$

F parallel to $\vec{n}$ yields greatest power output

Now the idea is to vary the field and the direction of the surface:

$$\iint_S F \cdot \vec{n} \, d\sigma = \iint_R F \cdot \vec{n} \left\| \frac{\partial x}{\partial t} \times \frac{\partial x}{\partial u} \right\| dt\,du$$

where $\vec{n}$ is the "outward" unit normal to surface at any given point.

Note: $\vec{n}\left\|\frac{\partial x}{\partial t}\times\frac{\partial x}{\partial u}\right\| = \frac{\partial x}{\partial t}\times\frac{\partial x}{\partial u}$ by def'n

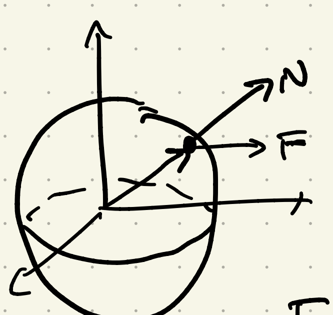
so $$\boxed{\iint_S F\cdot\vec{n}\,d\sigma = \iint_R F(x(t,u))\cdot\left(\frac{\partial x}{\partial t}\times\frac{\partial x}{\partial u}\right)dt\,du}$$

Often called "the flux of F through S"

Ex: $F(x,y) = (x,y,0)$, $S = \{x^2+y^2+z^2 = a^2\}$ w/ outward orientation

$$N(\varphi,\theta) = \frac{\partial x}{\partial \varphi}\times\frac{\partial x}{\partial \theta} = a\sin\varphi\, X(\varphi,\theta)$$

since $\varphi\in[0,\pi]$, $N(\varphi,\theta)$ points in the direction of the position vector $X(\varphi,\theta)$ $(\sin\varphi\geq 0)$



$$X(\varphi,\theta) = (a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi)$$

$$F(X(\varphi,\theta)) = (a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ 0)$$

$$F(X(\varphi,\theta))\cdot N(\varphi,\theta)$$
$$=(a\sin\varphi)(a^2\sin^2\varphi) = a^3\sin^3\varphi$$

$$\iint_S F\cdot\vec{n}\,d\sigma = a^3\int_0^{2\pi}\int_0^{\pi}\sin^3\varphi\,d\varphi\,d\theta$$

$$= 2\pi a^3\int_0^{\pi}\sin^3\varphi\,d\varphi$$

$$= 2\pi a^3\left(-\cos\varphi+\frac{1}{3}\cos^3\varphi\right)\Big|_0^{\pi} = 2\pi a^3\,\frac{4}{3} = \frac{8\pi a^3}{3}$$

$$\left(\begin{aligned}\int \sin^3\varphi \, d\varphi &= \int (1-\cos^2\varphi)\sin\varphi \, d\varphi \\ &= \int \sin\varphi - \int \cos^2\varphi \sin\varphi \\ &= -\cos\varphi + \tfrac{1}{3}\cos^3(\varphi)\end{aligned}\right)$$

<u>Note</u>: if your parametrization gives the opposite $N(t,u)$ of the one you want, you can just compute the integral and flip the sign at the end.