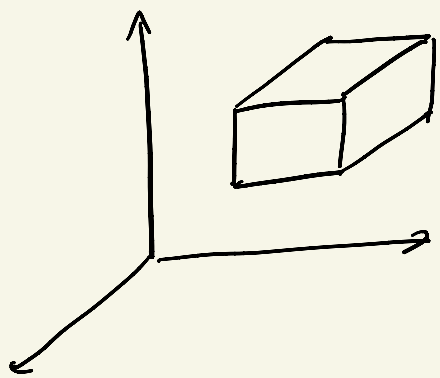
Triple Integrals

The ideas are largely the same as in the two-variable case.

$$R = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$$



$$\mathrm{vol}(R) = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$$

Thm: Suppose A is a region described by

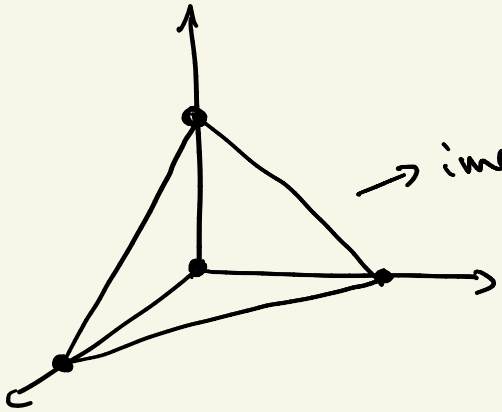
$$a \le x \le b, \quad g_1(x) \le y \le g_2(x),$$
$$\text{and} \quad h_1(x,y) \le z \le h_2(x,y).$$

Let f be continuous on A. Then

$$\iiint_A f = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left(\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, dz \right) dy \right] dx$$

<u>Ex:</u> Consider the tetrahedron T with vertices

$(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$



$\rightarrow$ imagine this as a solid

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{c} \overbrace{0 \le z \le 1-x-y}^{\text{why?}} \\ 0 \le y \le 1-x \\ 0 \le x \le 1 \end{array} \right\}$$

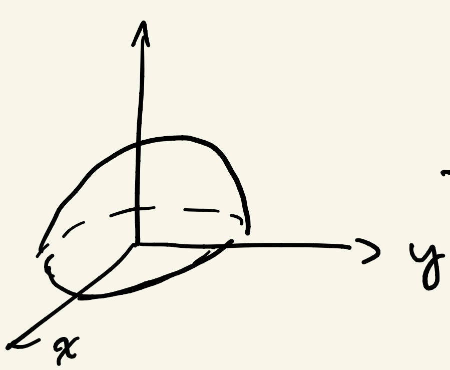
so if f is a function defined on T,

$$\iiint_T f = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f \, dz\, dy\, dx$$

$$\mathrm{vol}(T) = \iiint 1 \, dz\, dy\, dx = \frac{1}{6}.$$

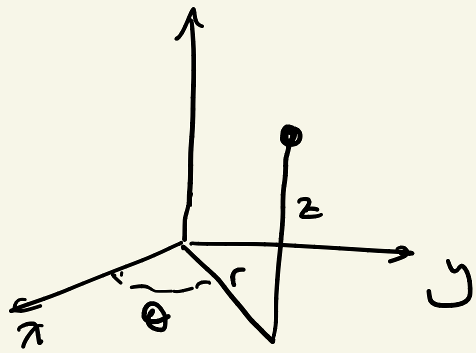
<u>Ex:</u> what region is described by

$0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$, $0 \le z \le \sqrt{1-x^2-y^2}$?

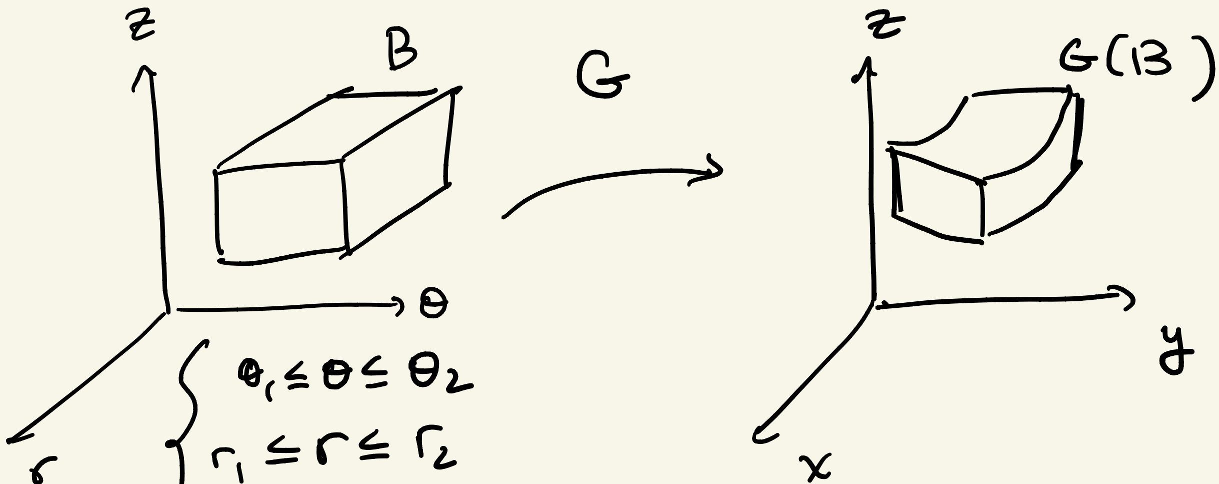


Cylindrical coordinates

$$G: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



point is specified by a radius, angle, then height.



$$\begin{cases} \theta_1 \le \theta \le \theta_2 \\ r_1 \le r \le r_2 \\ z_1 \le z \le z_2 \end{cases}$$

The volume of $G(B)$ is

$$(z_2 - z_1)\left(\frac{r_2^2 - r_1^2}{2}\right)(\theta_2 - \theta_1)$$

$$= \quad \bar{r}\,(z_2 - z_1)(r_2 - r_1)(\theta_2 - \theta_1)$$

<u>Thm:</u> Let A be the region in (x,y,z)-space whose cylindrical coordinates satisfy

$$a \le \theta \le b \quad (\text{and } b \le a + 2\pi)$$
$$0 \le g_1(\theta) \le r \le g_2(\theta)$$
$$h_1(\theta, r) \le z \le h_2(\theta, r)$$

Let $f^*(\theta, r, z) = f(r\cos\theta, r\sin\theta, z)$

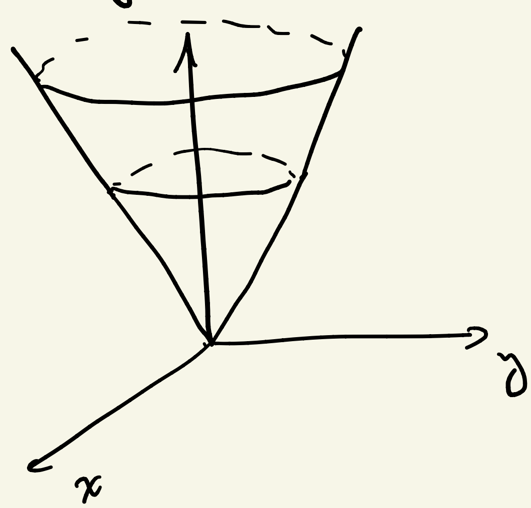
Then
$$\iiint_A f = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\theta,r)}^{h_2(\theta,r)} f^*(r,\theta,z)\, r\, dz\, dr\, d\theta.$$

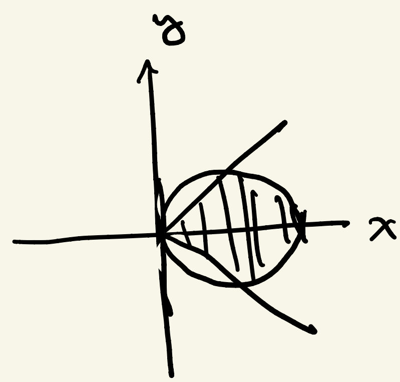
this guy again!

<u>Ex 1:</u> Find the mass of a solid bounded by

$$-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}, \quad r = \cos\theta, \quad z = 0, \text{ and } z = r$$

with density $f^*(z, r, \theta) = 3r$.

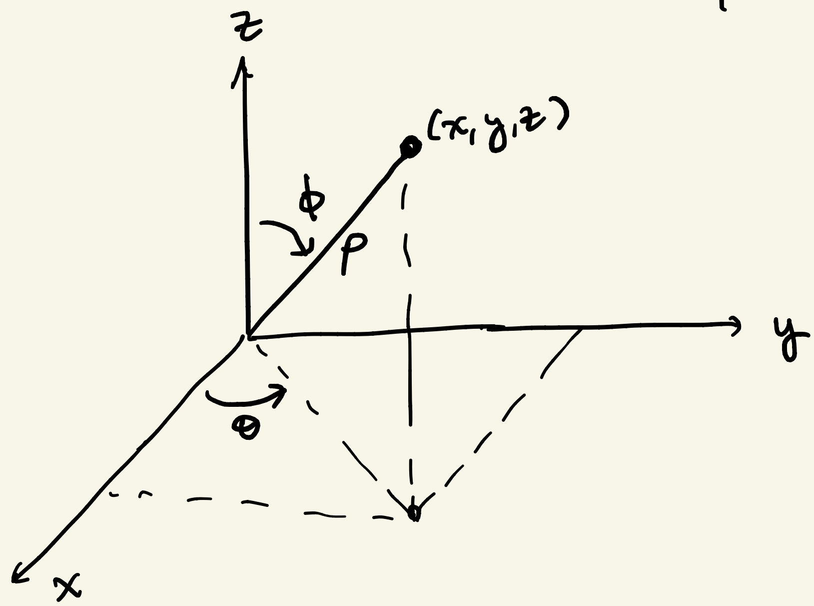




$(x - \frac{1}{2}) + y^2 \le \frac{1}{4}$

$$\int_{-\pi/3}^{\pi/3} \int_0^{\cos\theta} \int_0^{r} 3r \; r\cos\theta \, dz\, dr\, d\theta = \ldots$$

<u>Spherical coords</u> We can also specify a pt. in 3-space by two angles θ, ϕ and the distance ρ to the origin.



$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\boxed{\begin{aligned} z &= \rho\cos\phi \\ y &= \rho\sin\phi\sin\theta \\ x &= \rho\sin\phi\cos\theta \end{aligned}}$$

usually

$\theta \in [0, 2\pi]$

$\phi \in [0, \pi]$

$\rho \in \mathbb{R}_{\geq 0}$

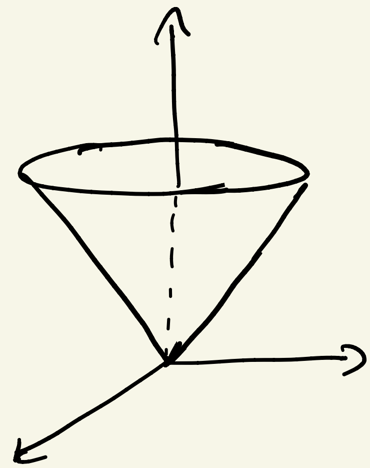
<u>Ex:</u> The sphere of radius a centered at (0,0,0) is

$$\rho = a.$$

The region $A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le a^2\}$

corresponds in (ρ, φ, θ) space to the set A^* defined by

$$\rho \le a.$$



Ex: $z = \sqrt{x^2+y^2}$

$$\Downarrow$$

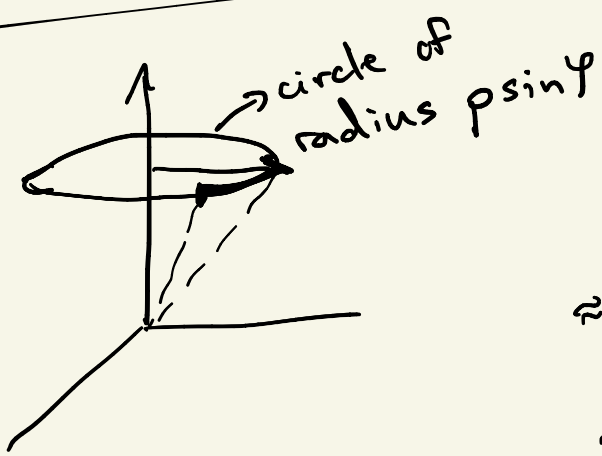
$$\rho\cos\varphi = \rho\sin\varphi$$

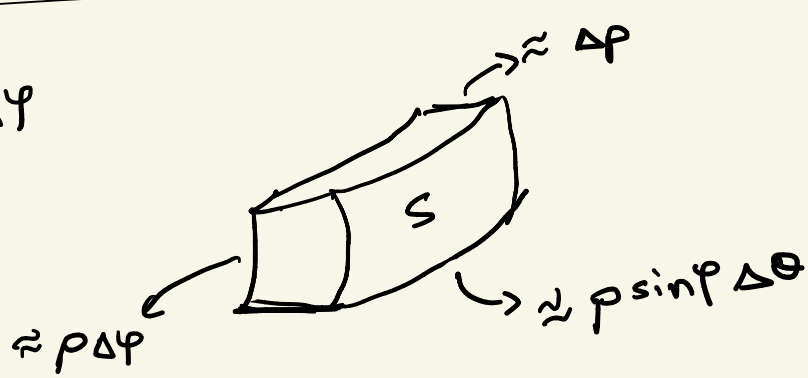
$$\Rightarrow \cos\varphi = \sin\varphi$$

$$\Rightarrow \varphi = \pi/4.$$

So this cone in (ρ, φ, θ) space is described simply by $\varphi = \pi/4$.

What if we wanted a solid cone?





$$\text{vol } S \approx \rho^2 \sin\varphi \, \Delta\rho\,\Delta\varphi\,\Delta\theta$$

$$\boxed{\iiint_A f(x,y,z)\,dz\,dy\,dx = \iiint_{A^*} f^*(\rho,\theta,\varphi)\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta}$$

$$\boxed{dV = dz\,dy\,dx = r\,dz\,dr\,d\theta = \rho^2 \sin\phi\,dz\,d\varphi\,d\theta}$$

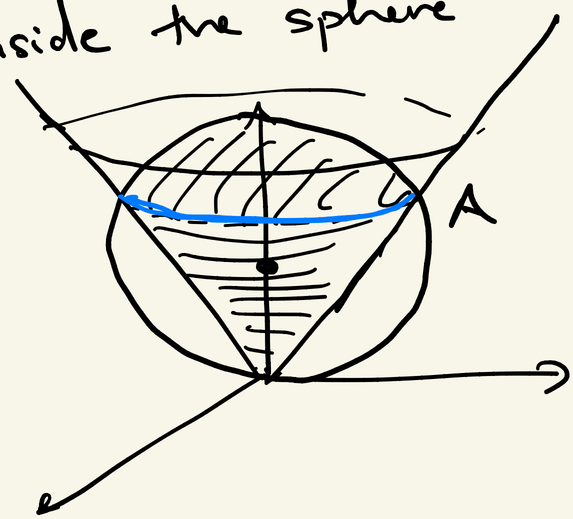
"volume elements" in the three coordinate systems

Ex: Find the volume of above the cone ① $z^2 = x^2+y^2$ and inside the sphere

② $x^2+y^2+z^2 = z$.

$\Updownarrow$

$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4}$



② becomes $\rho^2 = \rho\cos\varphi$

$\Rightarrow \rho = \cos\varphi$

① becomes $\varphi = \pi/4$

The region is the set of pts s.t.

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi/4$

$0 \le \rho \le \cos\varphi$

so $\text{vol}(A) = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3}{3}\sin\varphi \Big|_{\rho=0}^{\rho=\cos\varphi} d\varphi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} \cos^3\varphi \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} -\frac{\cos^4\varphi}{4}\Big|_{\varphi=0}^{\varphi=\pi/4} d\theta \qquad \left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4}$$

$$= \frac{-1}{12}\int_0^{2\pi}\left(\frac{1}{4}-1\right) d\theta = \left(\frac{-1}{12}\right)\left(-\frac{3}{4}\right)2\pi$$

$$= \frac{1}{8}\pi.$$

Thm: Let A be the region in $\mathbb{R}^3$ whose spherical coordinates satisfy

$$\begin{cases} a \le \theta \le b \\ g_1(\theta) \le \varphi \le g_2(\theta) \\ h_1(\theta,\varphi) \le \rho \le h_2(\theta,\varphi) \end{cases}$$

where g_1, g_2, h_1, h_2 are smooth functions, $0 \le b-a \le 2\pi$, and $0 \le g_1(\theta) \le g_2(\theta) \le \pi$.

Then

$$\iiint_A f = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\varphi,\theta)}^{h_2(\varphi,\theta)} f^*(\theta,\varphi,\rho)\,\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta.$$