

Conservation of Energy

Let $U \subset \mathbb{R}^n$ be an open set. A **vector field** F on U is a map $F : U \rightarrow \mathbb{R}^n$. So F maps points to vectors with the same dimension. One thinks of vector fields as a field of arrows attached to each point of U .

A very simple example of a vector field is the gravitational field

$$G(x, y, z) = (0, 0, -g).$$

This is really an approximation of the field if we're close enough to Earth's surface. One imagines a bunch of arrows pointing downward. If we introduce a point mass m into the field, the mass feels a force of $mG = (0, 0, -gm)$.

If F is a vector field and there exists some (scalar-valued) function f such that

$$\text{grad } f = F$$

then f is said to be **conservative**. The function f is called a **potential** function for F . For technical reasons in physics, sometimes the definition uses $-\text{grad } f$ instead, and we note that $-\text{grad } f = \text{grad}(-f)$, so there's no real difference.

Suppose a particle of mass m moves in U on which a field of forces $F : U \rightarrow \mathbb{R}^3$ is defined. That is, $F(x, y, z)$ is the force the particle is experiencing if it is positioned at (x, y, z) . Let $C(t)$ be a parametrization of the particle's path. Then Newton's law says that the force acting on the particle is equal to the mass times its acceleration:

$$F(C(t)) = mC''(t).$$

The **kinetic energy** of a particle is

$$\frac{1}{2}m\|C'(t)\|^2.$$

Conservation Law. Suppose that F is a conservative vector field and ψ satisfies $F = -\text{grad } \psi$. That is, ψ is the potential energy of the system. Suppose a particle of mass m moves (according to Newton's law) where this field is defined and that the field is the only force acting on the particle. Then the sum of potential and kinetic energy is constant.

We differentiate

$$\psi(C(t)) + \frac{1}{2}m\|C'(t)\|^2$$

with respect to t to obtain

$$\text{grad } \psi(C(t)) \cdot C'(t) + mC'(t) \cdot C''(t).$$

Applying Newton's law, this becomes

$$\text{grad } \psi(C(t)) \cdot C'(t) + C'(t) \cdot F(C(t)).$$

Finally, using $F = -\text{grad } \psi$, we see these terms cancel each other out.

Potential Functions

So far, we have seen scalar-valued functions, curves in space, and vector fields. Do not confuse these notions, warns Lang.

Are potentials unique? The role of a potential is somewhat similar to the role of an antiderivative in single variable calculus. There we learned an antiderivative is unique up to a constant. That is, if F and G are two antiderivatives of f , then $F = G + C$ for some constant C . Well, there's a slight technicality. Consider the following functions

$$g(x) = 1/x$$

and

$$f(x) = \begin{cases} 1/x + 2, & x > 0 \\ 1/x - 3, & x < 0. \end{cases}$$

Then f and g have the same derivative, but they do not differ by a constant. The issue here is that the domain of f and g , which is $\mathbb{R} \setminus \{0\}$, is not **connected**. So the more precise statement is that antiderivatives are unique up to a constant when the domain is connected.

The definition we'll take is the following: an open subset $U \subset \mathbb{R}^n$ is **connected** if between any two points in U , there is a differentiable path that joins them. More precisely, if P and Q lie in U , then there is a differentiable curve $X(t)$ such that $X(t_0) = P$ and $X(t_1) = Q$ for some t_0, t_1 .

Theorem. If U is an open connected subset of \mathbb{R}^n and f, g are differentiable with $\text{grad } f(X) = \text{grad } g(X)$ for each $X \in U$, then there exists a constant k such that

$$f(X) = g(X) + k$$

for all $X \in U$.

To prove this, let $\varphi(X) = f(X) - g(X)$. Then $\text{grad } \varphi(X) = 0$ for all $X \in U$. Now let $P, Q \in U$. There is some differentiable curve $X(t)$ connecting P to Q . On the other hand,

$$\frac{d}{dt}\varphi(X(t)) = \text{grad } \varphi(X(t)) \cdot X'(t).$$

But $\text{grad } \varphi(X) = 0$ for all $X \in U$. Thus this derivative is 0, meaning φ is constant on the path $X(t)$, which implies $\varphi(P) = \varphi(Q)$. This means φ is constant on U .

In the two variable case, a vector field takes the form

$$F(x, y) = (f(x, y), g(x, y)).$$

A potential function φ would then (by definition) satisfy the equations

$$\frac{\partial \varphi}{\partial x} = f, \quad \frac{\partial \varphi}{\partial y} = g.$$

So if we have this potential, then

$$\frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial x} \right) = \frac{\partial f}{\partial y}.$$

We also have by the other equation that

$$\frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial y} \right) = \frac{\partial g}{\partial x}.$$

On the other hand, our theorem about mixed derivatives says that $D_1 D_2 = D_2 D_1$. Thus we come to a necessary condition for a potential to exist:

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y} (*).$$

This gives a quick test to determine whether a vector field *does not* admit a potential. If the above equation doesn't hold, then there is no possibility for a potential to exist. The converse is only true under certain conditions. In other words, if you check the above condition and it holds, it's still not certain that a potential exists.

Example. Let $F(x, y) = (x^2 y, \sin(xy))$. We can verify that F does not admit a potential.

Local Existence of Potential Functions

Theorem. (two-variable case) Let f and g be differentiable real valued functions on an open set U of the plane. If U is either the entire plane or a rectangle and

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y},$$

then the vector field $F(x, y) = (f(x, y), g(x, y))$ has a potential.

Example. Let $F = (2xy, x^2 + 3y^2)$ on $U = \mathbb{R}^2$. If we let f and g be the first and second component functions respectively, then $D_1 g = 2x = D_2 f$. So the above theorem asserts a potential exists. We can find it by integration. A potential φ must satisfy

$$\frac{\partial \varphi}{\partial x} = 2xy.$$

We can integrate with respect to x to get

$$\varphi = x^2 y + u(y)$$

where u is some function of y . We know also that

$$\frac{\partial \varphi}{\partial y} = x^2 + 3y^2 = x^2 + u'(y).$$

This implies $u'(y) = 3y^2$. So we can for instance take $u(y) = y^3$ and let

$$\varphi(x, y) = x^2 y + y^3.$$

We see φ is a potential for F .

Theorem. Suppose $F = (f_1, f_2, f_3)$ is a vector field defined on an open rectangular box

with differentiable components that have continuous partial derivatives. If

$$D_i f_j = D_j f_i$$

for all indices i and j , then F has a potential function. The above condition means $D_1 f_2 = D_2 f_1$, $D_1 f_3 = D_3 f_1$, and $D_2 f_3 = D_3 f_2$. The same theorem holds in higher dimensions.

Example. Find a potential for the vector field (defined on 3-space)

$$F(x, y, z) = (y \cos(xy), x \cos(xy) + 2yz^3, 3y^2z^2).$$

If you check the conditions on the derivatives as stated in the theorem, you'll see that they hold. So we integrate the first component to see that

$$\varphi(x, y, z) = \sin(xy) + \psi(y, z)$$

where $\psi(y, z)$ is some function yet to be determined.

Functions Depending Only on the Distance from the Origin

Some functions only depend on the distance to the origin. The easiest example of such a function is

$$r(x, y) = \sqrt{x^2 + y^2},$$

which is the function that maps a point to its distance from the origin. In \mathbb{R}^3 the corresponding function would be $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. In \mathbb{R}^n it would be $r(x_1, \dots, x_n) = \sqrt{x_1^2 + \dots + x_n^2}$. In the two-variable case, we have

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}.$$

A similar computation shows $\partial r / \partial y = y/r$, so that

$$\text{grad } r = \left(\frac{x}{r}, \frac{y}{r} \right) = \frac{1}{r}(x, y).$$

If we set $X = (x, y)$, then this reads

$$\text{grad } r = \frac{1}{r}X.$$

The same computation works in n dimensions, so the above holds in general.

Example. Let $f(x, y) = \sin(r) = \sin(\sqrt{x^2 + y^2})$. Then

$$\frac{\partial f}{\partial x} = \cos(r) \frac{x}{\sqrt{x^2 + y^2}} = \frac{\cos(r)}{r}x, \quad \frac{\partial f}{\partial y} = \cos(r) \frac{y}{\sqrt{x^2 + y^2}} = \frac{\cos(r)}{r}y.$$

So we see that

$$\text{grad } f(X) = \frac{\cos(r)}{r}X.$$

In general, a function depending only on the radius can be written $f = g(r)$, where $g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$. Bear in mind that r itself is still a function of (x_1, \dots, x_n) , so $f = g(r)$ is a function is a function from \mathbb{R}^n to \mathbb{R} . Observe that (take $n = 3$ for concreteness)

$$\frac{\partial f}{\partial x} = g'(r) \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{g'(r)}{r} x.$$

Similar computations hold for D_2 and D_3 , and so we have

$$\text{grad } f(X) = \frac{g'(r)}{r} X.$$

Gravity

Newton's law of gravitation says that the gravitational force between two bodies is given by

$$\frac{Gm_1m_2}{r^2}.$$

The equal and opposite forces point towards each other. Imagine that m_1 is positioned at the origin and m_2 elsewhere. The position vector X of m_2 points at m_2 . If we want to model the force that m_2 is feeling, we need a vector pointing towards the origin. $-X$ does this. It is desirable to work with unit vectors, since for a unit vector A , $\|cA\| = |c|\|A\|$. $\|X\| = r$, so we have

$$F(X) = \frac{-Gm_1m_2}{r^2} \frac{X}{r} = \frac{-Gm_1m_2}{r^3} X$$

as the force F acting on m_2 . To find a potential for f , we use the above formula and set

$$\frac{g'(r)}{r} = \frac{-Gm_1m_2}{r^3},$$

which after solving gives

$$g(r) = \frac{Gm_1m_2}{r}$$

as a potential. The potential energy of m_2 is then

$$\psi = -g(r) = \frac{-Gm_1m_2}{r}.$$

A Special Vector Field

Let

$$G(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

To get a sense of what this field looks like, let's use polar coordinates. Fix a radius r and vary θ . Substituting $x = r \cos \theta$ and $y = r \sin \theta$, we have

$$G(x, y) = \frac{1}{r} (-\sin \theta, \cos \theta).$$

On the other hand, the velocity of $C(t) = (r \cos(t), r \sin(t))$ is $C'(t) = (-r \sin(t), r \cos(t))$, so the velocity of this circular trajectory is parallel to $G(x, y)$. So our vector field circulates

around the origin, and the magnitudes shrink as the radius increases. For small radius, the magnitude is large, and it blows up as we approach the origin, where the vector field is undefined.

We note that $D_2f = D_1g$ is satisfied for this vector field. So on any rectangle not containing the origin, we can find a potential function. One can verify that

$$\varphi(x, y) = \arctan(y/x)$$

is a potential defined on any rectangle that doesn't meet the line $x = 0$. Geometrically, this is saying that potential is given by the angle of the point.

In fact, if we delete a small sector near the positive x axis and define $\varphi(x, y) = \theta$, this is a potential for G . Here, θ ranges in $[0, 2\pi - c]$ for any small $c > 0$.

We'll see later why one cannot define a potential on all of $\mathbb{R}^2 \setminus \{0\}$.