## Points in 2-space, 3-space, and Beyond

In the same way that we can specify a point in the plane with two numbers, we can specify a point in space with three numbers $(x, y, z)$. In general, in $n$-space ($\mathbb{R}^n$), we can specify a point with a list of $n$ numbers $(x_1, \ldots, x_n)$.

Given two points in $\mathbb{R}^3$, we can define addition on them by adding corresponding coordinates:

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) := (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

In general,

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) := (a_1 + b_1, \ldots, a_n + b_n).$$

**Example.** Let $A = (2, 3)$, $B = (-1, 1)$. Then $A + B = (1, 4)$. The figure looks like a parallelogram.

**Example.** Let $A = (3, 1)$, $B = (1, 2)$. Then $A + B = (4, 3)$. We obtain a parallelogram again. This is always the case. Starting from the origin $O = (0, 0)$, we obtain $B$ by moving 1 unit right and then 2 units up. We get $A + B$ by first moving 3 to the right, then 1 up, and then repeating the same movement we did from the origin to $B$. In other words, the segment connecting $O$ to $B$ and the one connecting $A$ to $A+B$ are equal length and parallel. Similarly, the segments from $O$ to $A$ and $B$ to $B + A = A + B$ will also be equal length and parallel.

We have some not-so-surprising properties of point addition

- $(A + B) + C = A + (B + C)$
- $A + B = B + A$
- $O + A = A + O = A$
- $A + (-A) = O$

where $O = (0, \ldots, 0)$ and $-A = (-a_1, \ldots, -a_n)$. We note that $A \mapsto -A$ corresponds to reflection about the origin.

We can also multiply (or *scale*) a point $A = (a_1, \ldots, a_n)$ by a number $c$, yielding a point

$$cA = (ca_1, \ldots, ca_n).$$

For example, if $A = (2, -1, 5)$ and $c = 7$, then $cA = (14, -7, 35)$. We again have some easy properties:

- $c(A + B) = cA + cB$
- $(c_1 + c_2)A = c_1A + c_2A$
- $(c_1c_2)A = c_1(c_2A)$.

We should comment on the geometric meaning of scaling by a number $c$. Let $A = (1, 2)$ and $c = 3$. Then $cA = (3, 6)$. We see that the effect of multiplying by 3 is to stretch the point $A$ away from the origin by a factor of 3. If we set $c = 1/2$, this shrinks $A$ in towards the origin. If we draw a segment from the origin to $A$, in the former case, scaling by $c = 3$ multiplies the length by 3, and scaling by $c = 1/2$ cuts the length in half.

## Vectors

The discussion above leads us naturally to vectors. Given two points $A$ and $B$, we can define a **located vector** as an ordered pair of points $(A, B)$, which is more often written $\overrightarrow{AB}$. We think of this as an arrow connecting $A$ and $B$, pointing towards $B$. Two located vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be **equivalent** if $B - A = D - C$. We always have that $\overrightarrow{AB}$ is equivalent to $\overrightarrow{O(B-A)}$. This is actually the unique vector starting at the origin that is equivalent to $\overrightarrow{AB}$.

$\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are said to be **parallel** if for some $c \neq 0$, we have $A - B = c(Q - P)$. If $c > 0$ we say the vectors have the *same direction*, and if $c < 0$, we say they have the *opposite direction*.

$\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are said to be **perpendicular** if $B - A$ and $Q - P$ are perpendicular in the usual geometric sense.

A located vector starting from the origin is completely determined by its endpoint. So an $n$-tuple will be called either a point or a **vector** depending on the context and interpretation.

## The Dot Product

If $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$, their **dot product** is

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Note

Some useful properties are

1. $A \cdot B = B \cdot A$
2. $A \cdot (B + C) = A \cdot B + A \cdot C = (B + C) \cdot A$
3. If $x$ is a number, $(xA) \cdot B = x(A \cdot B)$, $A \cdot (xB) = x(A \cdot B)$
4. If $A = O$, then $A \cdot A = 0$. Otherwise, $A \cdot A > 0$.

Two vectors $A$ and $B$ are said to be **perpendicular** or **orthogonal** if $A \cdot B = 0$. For the plane and $\mathbb{R}^3$, we will see that this definition agrees with our previous and more geometric definition of perpendicular.

The **norm** or **magnitude** (or **length**) $\|A\|$ of a vector $A = (a_1, \ldots, a_n)$ is

$$\|A\| = \sqrt{A \cdot A} = \sqrt{a_1^2 + \cdots + a_n^2}.$$

Note that $\|-A\| = \|A\|$. More generally, for any number $c$, we have $\|cA\| = |c|\|A\|$. For two points $A, B$, the **distance** between them is

$$\|A - B\| = \sqrt{(A - B) \cdot (A - B)}.$$

A vector $E$ is a **unit vector** if $\|E\| = 1$. Dividing a nonzero vector by its norm always yields a unit vector, since

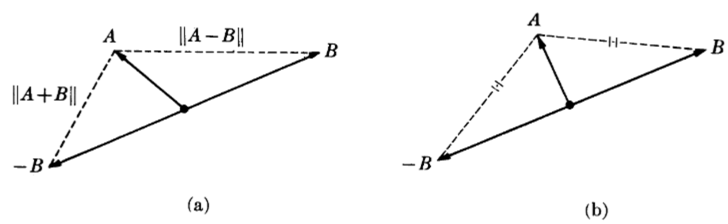
$$\left\| \frac{A}{\|A\|} \right\| = \frac{1}{\|A\|} \|A\| = 1.$$

Two nonzero vectors $A$ and $B$ have the **same direction** if there is some $c > 0$ such that $cA = B$. So, for instance, $A/\|A\|$ is a unit vector in the same direction as $A$.

## Perpendicularity, Angle Between Vectors

We have two notions of "perpendicular" floating around. One says $A$ and $B$ are perpendicular if $A \cdot B = 0$. The other is the more familiar notion of $A$ and $B$ forming a right angle. Suppose that $A$ and $B$ lie in the plane. We can convince ourselves that $A$ and $B$ form a right angle precisely when

$$\|A - B\| = \|A + B\|.$$




If we accept this, then the equivalence of our two definitions of perpendicularity will follow from

$$\|A + B\| = \|A - B\| \iff A \cdot B = 0.$$

To prove this, observe that

$$\begin{aligned} \|A + B\| = \|A - B\| &\iff \|A + B\|^2 = \|A - B\|^2 \\ &\iff A \cdot A + 2A \cdot B + B \cdot B = A \cdot A - 2A \cdot B + B \cdot B \\ &\iff 4A \cdot B = 0 \\ &\iff A \cdot B = 0. \end{aligned}$$

Suppose again that we have two nonzero vectors $A$ and $B$ in the plane, located at the origin. If we move along the line through $\overrightarrow{OB}$, there will be some point $P$ on this line such that $\overrightarrow{PA}$ is perpendicular to $\overrightarrow{OB}$. Then $P = cB$ for some number $c$. Then we have $(A - P) \cdot B = (A - cB) \cdot B = 0$, which is to say

$$A \cdot B - cB \cdot B = 0,$$

so that

$$c = \frac{A \cdot B}{B \cdot B}.$$

Conversely, we see that

$$\left(A - \frac{A \cdot B}{B \cdot B} B\right) \cdot B = A \cdot B - A \cdot B = 0.$$

Thus, this is the unique number $c$ that makes $A - cB$ perpendicular to $B$. This number $c$

is called the **component** of $A$ along $B$. If we do a little plane geometry, we see that

$$\cos\theta = \frac{c\|B\|}{\|A\|},$$

which can be rewritten as

$$\|A\|\|B\|\cos\theta = A \cdot B.$$

The **projection** of $A$ onto $B$ is

$$\frac{A \cdot B}{B \cdot B} B.$$

Note that if $D = cB$, then

$$\begin{aligned} \frac{A \cdot D}{D \cdot D} D &= \frac{A \cdot cB}{cB \cdot cB} cB \\ &= \frac{A \cdot B}{B \cdot B} B, \end{aligned}$$

so the projection of $A$ onto $B$ doesn't strictly depend on $B$; projecting onto any multiple of $B$ will yield the same vector (the component, however, *does* up to a sign).

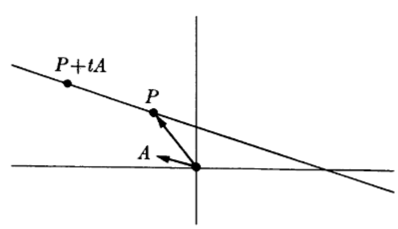
Note also that if $B$ is a unit vector, the component simplifies to $A \cdot B$, and the projection of $A$ onto $B$ simplifies to $(A \cdot B)B$.

## Parametric Lines

Given a direction vector $A$ and a point $P$, the **parametric line** in the direction of $A$ passing through $P$ is given by

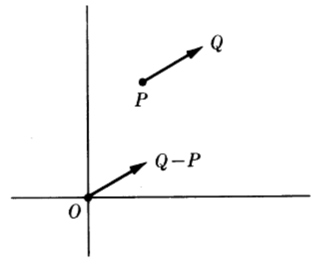
$$X(t) = P + tA,$$

where $t$ ranges in $\mathbb{R}$. One can think of this as the position $X$ of a particle or bug as it travels with the passing of time $t$. $X(t)$ is sometimes called the **position vector** of the particle/bug. The position vector is a vector located at the origin, terminating at the position of the bug. The figure below illustrates how the parametrization works: we start $P$, and as $t$ varies, we shift $P$ by a multiple of $A$. As $t$ ranges over all of $\mathbb{R}$, this traces out a line.




Given two points $P$ and $Q$, we can parametrize the line segment between them as

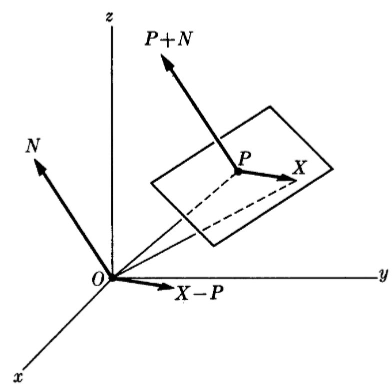
$$X(t) = P + t(Q - P), \ 0 \leq t \leq 1.$$




## Planes

A plane $M$ in $\mathbb{R}^3$ is determined by two pieces of data: a point $P$ lying on the plane, and a vector $N$ perpendicular to the plane. If we walk from $P$ to some other point $X$ also on the plane, we must have that $X - P$ is perpendicular to $N$, otherwise $X$ won't lie on the plane $M$. So the plane is the set of points $X$ satisfying

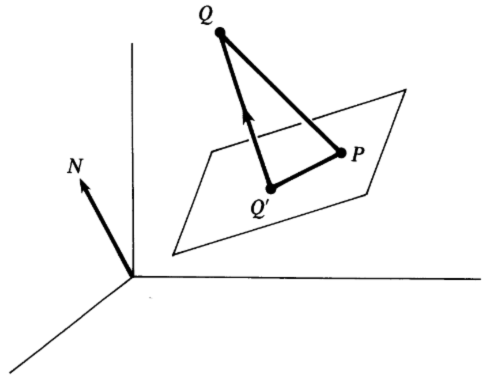
$$N \cdot (X - P) = 0.$$




Note that this gives a nice interpretation of the equation for a line in the plane $ax+by = c$. $(a, b)$ is a normal vector to the line! If $c = 0$, the equation becomes $(a, b) \cdot (x, y) = 0$, so we have a line through the origin, consisting of all vectors (located at the origin) perpendicular to $(a, b)$. Changing the value of $c$ yields a family of parallel lines.

Suppose that we have a plane passing through $P$ and perpendicular to $N$, and let $Q$ be some point not on the plane. How can we compute the (smallest) distance of $Q$ to the plane? The smallest distance corresponds to the length of the segment formed by drawing a perpendicular to the plane from the point $Q$. We can obtain this length by projecting $Q - P$ onto $N$ and taking the norm:

$$\text{length} = \left|(Q - P) \cdot \frac{N}{\|N\|}\right|$$




## The Cross Product

If $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are vectors in $\mathbb{R}^3$, their cross product $A \times B$ is the determinant

$$\begin{vmatrix} E_1 & E_2 & E_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

This is really more of mnemonic device than an actual definition, since the determinant is defined only for matrices with numerical entries.

$A \times B$ is perpendicular to both $A$ and $B$. We also have *anticommutativity*, meaning $B \times A = -(A \times B)$.

One can verify that $\|A \times B\|^2 = \|A\|^2 \|B\|^2 - (A \cdot B)^2$.

Using our geometric formula for the dot product, we have

$$\begin{aligned} \|A \times B\|^2 &= \|A\|^2 \|B\|^2 - \|A\|^2 \|B\|^2 \cos^2(\theta) \\ &= \|A\|^2 \|B\|^2 (1 - \cos^2(\theta)) \\ &= \|A\|^2 \|B\|^2 \sin^2(\theta). \end{aligned}$$

Taking square roots, we have

$$\|A \times B\| = \|A\| \|B\| |\sin(\theta)|.$$

So the magnitude of the cross product is the area of parallelogram spanned by $A$ and $B$.