Final Exam

Exercise 1. In this exercise, you will derive a formula for the distance between two lines $L_1(t) = A_1t + B_1$ and $L_2(t) = A_2t + B_2$ in $\mathbb{R}^3$ (that is, the minimal distance between any pair P, Q of points where P lies on L_1 and Q on L_2).

(a) Let $f : X \to \mathbb{R}$ be a function on some set X. Let $Y \subseteq X$ and suppose $y \in Y$ is a global minimum for f. That is,

$$f(y) \leq f(x)$$

for all $x \in X$. Show/explain why y is also a global minimum for $f|_Y : Y \to \mathbb{R}$, where $f|_Y$ is the restriction of f to Y. (Don't overthink this. This is simply a helpful observation for later.)

(b) Assume for simplicity that L_1 and L_2 are not parallel. Find equations (in terms of the A_i and B_i) for two parallel planes, one containing L_1 and the other containing L_2. In particular, note that there exist such planes. (Hint: there is a very natural expression for the normal vector.)

(c) Recall that the *projection* of a vector A onto a vector B is

$$\frac{A \cdot B}{B \cdot B} B.$$

Use this to find the distance between the two planes (this is the length of any perpendicular segment connecting the two planes). Your answer should be an expression involving A_1, A_2, B_1, and B_2.

(d) Observe that if two non-parallel lines L_1 and L_2 lie in parallel planes (as above), then there is a point P on L_1 and a point Q on L_2 such that the segment PQ is perpendicular to both planes. In light of part (a), explain why the distance you found in part (c) is indeed the distance between the lines L_1 and L_2.

Exercise 2. If we drag a line segment around in the plane, we sweep out some area (think of a roller paint brush). Similarly, if we drag a planar figure (such as a square or disk) around in 3-space, we obtain some kind of solid. The volume V of this solid is given by Guldin's formula, which states

$$V = \int_{t_0}^{t_1} A(t)\mathbf{n}(t) \cdot C'(t)dt,$$

where $C(t)$ is the curve traced out by the centroid/center of mass of the planar figure, $\mathbf{n}(t)$ is the unit normal of the planar figure, and $A(t)$ is the area of the planar figure. Note that $A(t)$ can depend on time, meaning the figure is allowed to change shape as it traverses through space. Note also that $\mathbf{n} \cdot C'(t)$ is simply the component of the velocity of the centroid of the figure along the direction perpendicular to the figure.

(a) Use this formula to derive the volume of a cone with a base of radius r and a height h. (Hint: a cone is obtained by dragging a disk upward at constant speed, while letting the radius shrink to 0 linearly with time. If we center the cone around the z-axis, the trajectory of the centroid is simply $C(t) = (0, 0, t)$, where $0 \leq t \leq h$).

(b) Now suppose that we skew the cone by letting the cross section A "drift" in the x and y directions, so that the centroid now traces out $C(t) = (k_1 t, k_2 t, t)$. What is the volume of the resulting solid?

Exercise 3. Recall that for any two vectors A and B,

$$|A \cdot B| \leq \|A\| \|B\|.$$

Also recall that

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

We also have seen that if $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Recall once again that the length of a curve is the integral of speed:

$$\int_a^b \|C'(t)\|dt.$$

Suppose now that $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field that is **bounded**. This means there is some number M such that $\|F(X)\| < M$ for all $X \in \mathbb{R}^3$. Use the above facts to show that for a curve $C(t)$ defined for $a \leq t \leq b$, we have

$$\left| \int_a^b F(C(t)) \cdot C'(t)dt \right| \leq M \operatorname{length}(C).$$

In other words, on a bounded vector field, the path integral $\int F \cdot dC$ will have a small value if the path is short.

Exercise 4. Let C be the curve of intersection of the cylinder $x^2 + y^2 = 2y$ and the plane $y = z$. Use Stokes' theorem to show that

$$\int_C y^2 \, dx + xy \, dy + xz \, dz = 0.$$

(Hint: think about the relation between the normal vector of the plane $y = z$ and the curl of the field $F = (y^2, xy, xz)$.)

Exercise 5. Suppose $v_1, \ldots, v_m \in V$ are linearly independent. Let $w \in V$. In this exercise, you will show that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1. \; (*)$$

(a) Let V_1, V_2 be subspaces of V. Using the equation

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim V_1 \cap V_2,$$

show that

$$\dim V_1 \geq \dim(V_1 + V_2) - \dim V_2.$$

(b) Show that

$$\operatorname{span}(v_1 + w, \ldots, v_m + w) + \operatorname{span}(w) \supseteq \operatorname{span}(v_1, \ldots, v_m).$$

(c) Recall that if U and W are subspaces such that $U \subseteq W$, then $\dim U \leq \dim W$. Note also that the span of a list containing just one vector has either dimension 1 or 0. Use these facts to show $(*)$.

Exercise 6. Give an example of $T \in \mathcal{L}(\mathbb{R}^4)$ such that $\operatorname{range} T = \operatorname{null} T$ (Hint: shift). Show, on the other hand, that there does *not* exist $T \in \mathcal{L}(\mathbb{R}^5)$ such that

$$\operatorname{range} T = \operatorname{null} T.$$