HW 8.

Exercise 1. Use Green's Theorem to compute the integral

$$\int_C y^2\ dx + x\ dy$$

where C is

- (a) the square with vertices $(0,0)$, $(0,2)$, $(2,0)$, $(2,2)$, oriented counterclockwise
- (b) the circle of radius 2 centered at the origin, oriented counterclockwise.

Exercise 2. Let A be a region, which is the interior of a closed curve C oriented counterclockwise. Show that the area of A is given by

- $\text{Area}(A) = \frac{1}{2}\int_C -y\ dx + x\ dy$
- $\text{Area}(A) = \int_C x\ dy$.

Exercise 3. Suppose that f satisfies Laplace's equation

$$\frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f = 0$$

everywhere on a region A which is the interior of a counterclockwise curve C. (The above equation could also be expressed as $D_1^2 f + D_2^2 f = 0$.) Show that

$$\int_C \left(\frac{\partial f}{\partial y}dx - \frac{\partial f}{\partial x}dy\right) = 0.$$

Exercise 4. Let $F = (p, q)$ be a vector field on the plane. Prove the divergence theorem (in two dimensions) by applying Green's theorem to the vector field $G = (-q, p)$.

Exercise 5. Find the volume of the region spanned by the following inequalities:

$$0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1 - x^2},\ 0 \leq z \leq \sqrt{1 - x^2 - y^2}.$$

Exercise 6. Find the volume of the cone

$$\sqrt{x^2 + y^2} \leq z \leq 1$$

via integration in spherical coordinates.

Exercise 7. Compute the integral

$$\iiint_A x^2\ dV$$

where A is the region inside the cylinder $x^2 + y^2 \leq a^2$ and between the planes $z = 0$ and $z = b$, where $b > 0$. (Note that $x^2 + y^2 \leq a^2$ describes a disk in the plane, but in 3-space it describes a cylinder since z can vary freely.) (Note: the identity $\cos^2\theta = (1 + \cos\theta)/2$ will likely be useful.)