HW 9.

Exercise 1. Fix some α with $0 < \alpha < \pi/2$ and some $h > 0$. Let

$$X(\theta, t) = (t \sin\alpha \cos\theta, t \sin\alpha \sin\theta, t\cos\alpha),$$

where $\theta \in [0, 2\pi]$ and $t \in [0, h \sec\alpha]$. This parametrizes a cone of height h. Compute the tangent vectors $\partial X/\partial\theta$ and $\partial X/\partial t$ as well as their cross product. Find the equation for the surface in rectangular/Cartesian coordinates.

Exercise 2. Fix $h, a > 0$ Let

$$X(t, \theta) = (at\cos\theta, at\sin\theta, t^2)$$

where $\theta \in [0, 2\pi]$ and $t \in [0, h]$. This parametrizes a paraboloid. Compute the tangent vectors $\partial X/\partial\theta$ and $\partial X/\partial t$ as well as their cross product. Find the equation for the surface in rectangular/Cartesian coordinates. What surface do we obtain if we replace t^2 with t in the z coordinate?

Exercise 3. Compute the surface area of the torus parametrized by

$$X(\theta, \varphi) = ((a + b\cos\varphi)\cos\theta, (a + b\cos\varphi)\sin\theta, b\sin\varphi)$$

where $0 \le \varphi < 2\pi$ and $0 \le \theta < 2\pi$. (Hint: the cross product computation is a little messy, but using the identity $\sin^2 x + \cos^2 x = 1$ a couple of times, you can simplify the expression for $\left\| \frac{\partial X}{\partial\varphi} \times \frac{\partial X}{\partial\theta} \right\|$ considerably).

Exercise 4. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Show that

$$\operatorname{curl}(\operatorname{grad}\varphi) = 0.$$

Exercise 5. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Show that $\operatorname{div}\operatorname{curl} F = 0$.

Exercise 6. Compute the integral

$$\iint_S \operatorname{curl} F \cdot \vec{n} d\sigma$$

where $F = (-y, x^2, z^3)$, and S is the surface

$$x^2 + y^2 + z^2 = 1, \ -\frac{1}{2} \le z \le 1.$$

[Hint: by Stokes' theorem, $\iint_S \operatorname{curl} F \cdot \vec{n} d\sigma$ of any two surfaces that share the same boundary (with the same orientation) will have the same value, so one can deform this surface (while holding the boundary fixed) to a much simpler one and compute the integral of $\operatorname{curl} F$ on the deformed surface instead.]

Exercise 7. Let C be a closed curve which is the boundary of a surface S. Let f and g be scalar-valued functions. Show that

$$\int_C (f \operatorname{grad} g) \cdot dC = \iint_S [(\operatorname{grad} f) \times (\operatorname{grad} g)] \cdot \mathbf{n} d\sigma.$$

Show consequently that

$$\int_C (f \operatorname{grad} g + g \operatorname{grad} f) \cdot dC = 0.$$