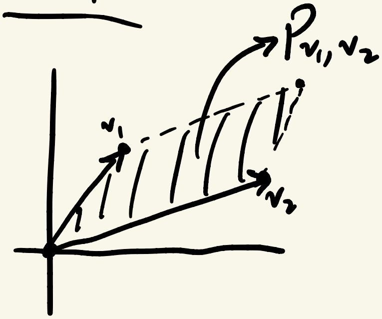
Determinants

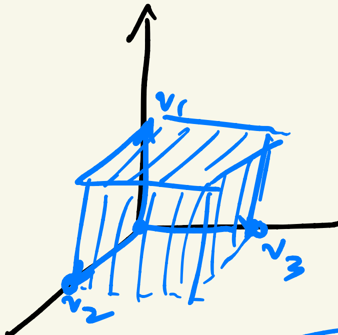
$v_1, v_2, \ldots, v_n \in \mathbb{R}^n$

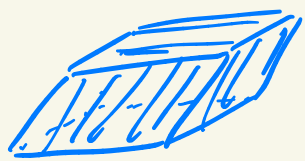
$$P_{v_1,\ldots,v_n} = \{ t_1 v_1 + \ldots + t_n v_n : 0 \le t_k \le 1,\ k = 1, \ldots, n \}$$

in $\mathbb{R}^2$



in $\mathbb{R}^3$





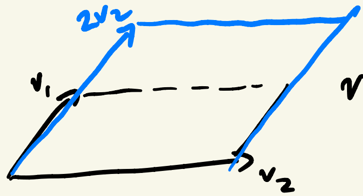
Aim to define a function $D(v_1, \ldots, v_n)$ that corresponds to the "volume" of $P_{v_1,\ldots,v_n}$

$$\det A = D(v_1, \ldots, v_n)$$

where $A = (\vec{v}_1 \ \ldots \ \vec{v}_n)$

Properties the determinant should have:

$D(\alpha v_1, v_2, \ldots, v_n)$?

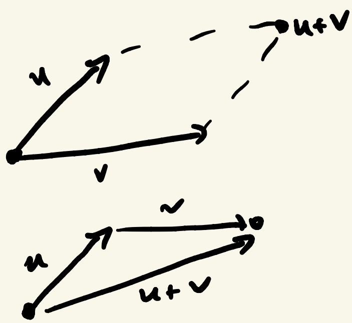


volume doubles

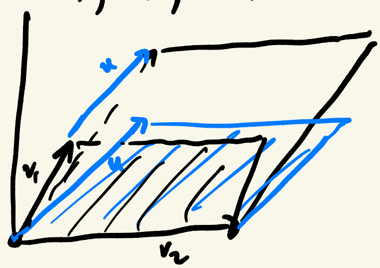
$D(\alpha v_1, v_2, \ldots, v_n) = \alpha D(v_1, v_2, \ldots, v_n)$

same for αv_i, not just v_1

$D(u + v_1, v_2, \ldots, v_n)$?



heights add together since projection is linear



area of parallelogram = base · height

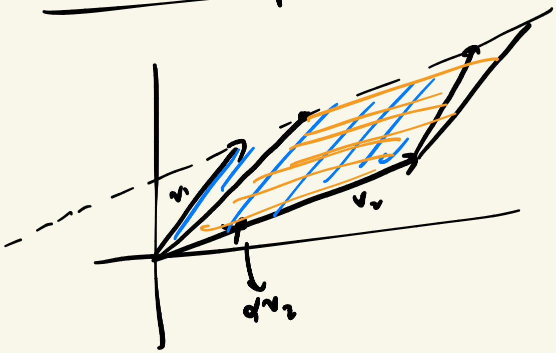
$D(u + v_1, v_2, \ldots, v_n)$

$= D(u, v_2, \ldots, v_n) + D(v_1, v_2, \ldots, v_n)$

So D ought to be linear in each argument (i.e. if we "freeze" all but one entry, the resulting map is linear)

here, we're allowing for negative heights

Column replacement



volume is unchanged; same base, same height

$$D(v_1, \ldots, \overset{j}{v_j + \alpha v_k}, \ldots, \overset{k}{v_k}, \ldots, v_n)$$

$$= D(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_n)$$

Antisymmetric

$$D(v_1, \ldots, \underset{j}{v_k}, \ldots, \underset{k}{v_j}, \ldots, v_n)$$

$$= -D(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_n)$$

interchanging entries flips sign

Follows from previous properties:

$$D(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_n)$$

$$= D(v_1, \ldots, v_j, \ldots, v_k - v_j, \ldots, v_n)$$

$$= D(v_1, \ldots, v_j + (v_k - v_j), \ldots, v_k - v_j, v_n)$$

$$= D(v_1, \ldots, v_k, \ldots, v_k - v_j, v_n)$$

$$= D(v_1, \ldots, v_k, \ldots, (v_k - v_j) - v_k, v_n)$$

$$= D(v_1, \ldots, v_k, \ldots, -v_j, \ldots, v_n) \qquad -v_j = (-1)v_j$$

$$= -D(v_1, \ldots, v_k, \ldots, v_j, \ldots, v_n)$$

Normalization property

$$D(\vec{e}_1, \dots, \vec{e}_n) = 1$$

i.e. $\det I = 1$.

The following properties will uniquely define the determinant:

1) Linearity in each col/entry

$$D(v_1, \dots, \alpha u_k + \beta v_k, \dots, v_n)$$

$$= \alpha D(v_1, \dots, u_k, \dots, v_n) + \beta D(v_1, \dots, v_k, \dots, v_n)$$

2) Antisymmetry

$$D(v_1, \dots, \overset{j}{v_k}, \dots, \overset{k}{v_j}, \dots, v_n) = -D(v_1, \dots, v_n)$$

3) $\operatorname{Det} I = 1$.

Section 3.2

Proposition 3.1: A = square matrix

1. If A has zero col, then $\det A = 0$
2. If A has two equal cols, then $\det A = 0$
3. If one col is a multiple of another, then $\det A = 0$
4. If the cols are linearly dependent, (i.e. A not invertible), then $\det A = 0$.

Pf: 1. scaling a 0 col by 0 changes nothing about A. But then $\det A = 0$ by scaling property.

2. exchange the two identical cols

$$\det A = -\det A$$

$$\Rightarrow \det A = 0$$

3. use 2. and linearity

4. Suppose first that

$$v_1 = \sum_{k=2}^{n} \alpha_k v_k$$

$$D(v_1, \dots, v_n) = D\left(\sum_{k=2}^{n} \alpha_k v_k, v_2, \dots, v_n\right)$$

$$= \sum_{k=2}^{n} \alpha_k D(v_k, v_2, \dots, v_n).$$

each term is 0. why?

General case: exchange v_k to sit in v_1. $\square$

Prop 3.2: Adding a linear comb. of <u>other</u> cols to a col doesn't change det. In particular, the det is preserved by type 3 col op (col replacement).

Pf: Fix $\vec{v}_k$. Suppose

$$\vec{u} = \sum_{j \neq k} \alpha_j \vec{v}_j .$$

Then by linearity

$$D(v_1, \ldots, v_k + u, \ldots, v_n)$$

$$= D(v_1, \ldots, v_k, \ldots, v_n) + \underbrace{D(v_1, \ldots, u, \ldots, v_n)}_{\substack{= 0 \text{ since cols} \\ \text{not lin indep.}}} . \square$$

Diagonal and triangular matrices

$A=(a_{i,j})$ **Def:** A is diagonal if $a_{j,k}=0$ when $j\neq k$.

$\mathrm{diag}(a_1,\dots,a_n)$ denotes the matrix

$$\det \mathrm{diag}(a_1,\dots,a_n) = a_1 \det \mathrm{diag}(1,a_2,\dots,a_n)$$

$$= \dots = a_1 a_2 \cdots a_n .$$

Def: $A=\{a_{i,j}\}_{i,j=1}^{n}$ (square) is called upper triangular if

$a_{j,k}=0$ for all $k<j$.

lower triangular if $a_{j,k}=0$ when $k>j$.

triangular if it is either LT or UT.

Prop: If $A=(a_{i,j})$ is triangular, then

$$\det A = a_{1,1} a_{2,2} \cdots a_{n,n}.$$

Pf: First suppose there is a 0 on diagonal.

Find the first 0 on diagonal.

$$\begin{array}{c} \begin{pmatrix} * & & & \vdots & & \\ & * & & \vdots & & \\ & & * & \vdots & & \\ & & & 0 & & \\ & & & 0 & \ddots & \\ & & & \vdots & & \ddots \\ & & & 0 & & \end{pmatrix} \\ \ \ k \end{array}$$

add multiple of first col to k-th col to clear first entry of k-th col.

Use 2nd col to clear 2nd entry of k-th col

$$\begin{array}{c} \downarrow \\ \begin{pmatrix} * & & & 0 & \\ & * & & 0 & \\ & & * & \vdots & \\ & & & \vdots & \\ & & & 0 & \end{pmatrix} \\ k \end{array} \Rightarrow \det A = 0$$

(col repl. preserves det)

Suppose no 0's on diag. Then same col ops as above yield

$$\begin{pmatrix} a_{1,1} & & 0 \\ & a_{2,2} & \\ 0 & & \ddots & \\ & & & a_{n,n} \end{pmatrix} \Rightarrow \det A = a_{1,1}\, a_{2,2} \cdots a_{k,k}.$$

$\square$

Computing det A for general A

- we still haven't defined det A, but we'll do this later

- <u>idea:</u> do column reduction on A (row red. on A^T) to obtain ech. form, keep track of ops. Only scaling and col. swap change value.

- If ech form of A^T does not have pivot in every col, then A is noninvertible, so det $A = 0$. If A inv, then result is triangular, so then det A = (product of diag)(correction factor)

Thm 3.4 $\det A = \det A^T$

Thm 3.5 $A, B \in M_{n \times n}$.

$$\det(AB) = \det(A)\det(B)$$

Lemma: For a square mat. A and elementary E,

$$\det(AE) = (\det A)(\det E).$$

Pf: AE corresponds to performing a col op.

if E is col exchange $\det E = -1$.

if E is col scaling by α, then $\det E = \alpha$.

(using that $\det I = 1$)

if E is col repl, then E is triangular with all 1's on diag $\Rightarrow \det E = 1$. $\square$

Repeated application of lemma gives

$$\det(AE_1 \dots E_N) = \det A \det E_1 \dots \det E_N$$

Obs: Any invertible A is a product of elementary matrices. why?

A is row. eq to ?

Pf of Thm 3.4

If E elem., $\det E = \det E^T$.

Note if A not inv, neither is A^T, so thm holds automatically. (rank A = rank A^T).

So suppose A inv.

$$A = E_1 \dots E_N$$

$$A^T = E_N^T \dots E_1^T$$

$$\det A = \det(E_1) \dots \det(E_N)$$

$$= \det(E_1^T) \dots \det(E_N^T)$$

$$= \det A^T. \square$$

Pf Thm 3.5: First suppose B invertible.

$$B = E_1 E_2 \dots E_N$$

$$\det(AB) = \det(A E_1 \dots E_N)$$

$$= \det(A) \det(E_1) \dots \det(E_N)$$

$$= \det(A) \det(B).$$

If B not inv, then AB noninv also, otherwise $AB = C \rightarrow$ inv, so $C^{-1}AB = I$

$\Rightarrow$ B left inv $\Rightarrow$ B inv since B is square.

So then thm is simply saying $0 = 0$. $\square$

Summary of properties

$\det A$ is defined for <u>square</u> matrices

$\det A = \det A^T$

1. det is linear in each row (each col)
2. interchanging two rows (or cols) flips sign
3. A triangular $\Rightarrow \det A$ is product of diagonal entries

   In particular, $\det I = 1$.

4. A has a 0 row (col) $\Rightarrow \det A = 0$
5. If A has two equal rows (or cols), $\det A = 0$.
6. $\det A \neq 0 \Longleftrightarrow A$ invertible
7. row replacement (col repl) preserves det.
8. $\det(AB) = (\det A)(\det B)$
9. $\det(\alpha A) = \alpha^n \det A$.

Formal Definition of det

$$A = (a_{j,k})_{j,k=1}^{n}, \quad \text{cols } \vec{v}_1, \ldots, \vec{v}_n$$

i.e. $$v_k = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{n,k} \end{pmatrix} = a_{1,k}\vec{e}_1 + \cdots + a_{n,k}\vec{e}_n$$

$$= \sum_{j=1}^{n} a_{j,k}\vec{e}_j$$

Using linearity in 1st col:

$$D(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = D\left(\sum_{j=1}^{n} a_{j,1}\vec{e}_j, \vec{v}_2, \ldots, \vec{v}_n\right)$$

$$= \sum_{j=1}^{n} a_{j,1} D(\vec{e}_j, \vec{v}_2, \ldots, \vec{v}_n)$$

Do this for each col of A

$$D(\vec{v}_1, \ldots, \vec{v}_n) = \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} \cdots \sum_{j_n=1}^{n} a_{j_1,1}\, a_{j_2,2} \cdots a_{j_n,n} D(e_{j_1}, \ldots, e_{j_n})$$

Big sum: n^n terms

A lot of the terms are 0, though

if any 2 indices among $j_1, \dots, j_n$ coincide,

$D(e_{j_1}, \dots, e_{j_n}) = 0$. why?

The only possibly nonzero terms are those for which $j_1, j_2, \dots, j_n$ are all different.

In other words, $j_1, j_2, \dots, j_n$ is some reordering of the numbers $1, \dots, n$.

Def: A function $f: A \to B$ is one-to-one if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$. i.e. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Def: f is onto if

$$f(A) = B.$$

i.e. $\forall b \in B$, there is some $a \in A$ s.t.

$$f(a) = b.$$

Def: A permutation of $\{1, 2, \dots, n\}$ is a $\underbrace{\text{one-to-one, onto}}_{\text{"bijection"}}$ function $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$

i.e. a permutation is a way of rearranging n objects that are ordered in a line (a "shuffle")

The set of permutations of $\{1, \dots, n\}$ is denoted $\mathrm{Perm}(n)$.

$$\begin{array}{c} \text{in} \\ \text{out} \end{array}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

$\mathrm{Perm}(n)$ has $n!$ elements.

So

$$D(\vec{v}_1, \dots, \vec{v}_n) = \sum_{\sigma \in \mathrm{Perm}(n)} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n} \cdot \underbrace{D(\vec{e}_{\sigma(1)}, \dots, \vec{e}_{\sigma(n)})}$$

$(e_{\sigma(1)}, \dots, e_{\sigma(n)})$ is either $+1$ or -1. why?

- This leads to the notion the <u>sign</u> of a permutation

<u>Def:</u> An <u>inversion</u> in a permutation σ is a pair (j,k) where $j<k$ and $\sigma(j) > \sigma(k)$.

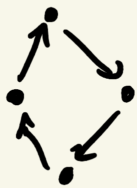
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \longrightarrow N(\sigma) = 3$$

<u>inversions:</u> $(1,3)$ $(2,3)$ $(2,4)$

Let $N(\sigma)$ denote the # of inversions

- Permutations can be broken into <u>cycles</u>.

$$1 \to \sigma(1) \to \sigma(\sigma(1)) \to \sigma(\sigma(\sigma(1)))$$
$$\to \ldots \text{ eventually repeats}$$



e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$

$$1 \to 3 \to 5 \qquad 2 \to 4$$

$$\sigma = (1 \; 3 \; 5)(2 \; 4)$$

cycle notation

A <u>transposition</u> is a permutation that exchanges two elements and leaves everything else fixed. In cycle notation,

$$\sigma = (n_1 \; n_2)$$

e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = (1 \; 3)$

note: $(1 \; 3 \; 5) = (1 \; 3)(3 \; 5)$

$$(n_1 \; n_2 \; \ldots \; n_k) = (n_1 \; n_2)(n_2 \; n_3) \cdots (n_{k-1} \; n_k)$$

"feed in number"

- Any $\sigma \in \mathrm{Perm}(n)$ can be written as $\sigma = \tau_1 \tau_2 \dots \tau_m$, where τ_i's are transpositions.

this decomposition isn't unique

e.g. $(1\ 2)(1\ 2)(1\ 2)(1\ 2) = id$

<u>Claim:</u> the # m will <u>always</u> be even or odd

i.e. $\sigma = \tau_1 \tau_2 \dots \tau_m$, $\sigma = t_1 t_2 \dots t_\ell$ transpositions $\Longrightarrow$ (m and ℓ even) OR (m and ℓ odd)

<u>Pf:</u> $\sigma = \tau_1 \tau_2 \dots \tau_k$

we show that the parity of k is the same as the parity of number of inversions $N(\sigma)$; either both even or both odd.

$$(2\ 5) = \underbrace{(2\ 3)(3\ 4)(4\ 5)(4\ 3)(3\ 2)}_{\text{adjacent transpositions}}$$

can always do this:

$(i\ \ i+d)$ = product of $2d-1$ transpositions

do this to each τ_i to get

$\sigma = t_1 t_2 \dots t_m$, each t_i is <u>adjacent</u>

Obs: if ρ is a permutation, and t is an adjacent transp (i.e. $t = (l \;\; l+1)$), then ρt has <u>one more</u> or <u>one less</u> inversion than ρ.

$$3, 4, \underbrace{1, 6,} 2, 5$$

$$\sigma = \tau_1 \tau_2 \cdots \tau_k = t_1 t_2 \cdots t_m$$

Note: $m = \sum_{i=1}^{k} (2 l_i + 1) = k + (\text{even } \#)$

so m and k have same parity

Let a be the # of inversion-increasing t_i's. b # of inversion-decr t_i's.

$$m = a + b$$

$$N(\sigma) = a - b$$

$\Rightarrow m - N(\sigma) = 2b$

$\Rightarrow m$ and $N(\sigma)$ have same parity

Thus k and $N(\sigma)$ have same parity, and parity of k doesn't depend on the particular decomposition.

<u>Def:</u> $\operatorname{sign} \sigma = (-1)^{N(\sigma)}$

<u>Def:</u> $D(v_1, \ldots, v_n) = \sum_{\sigma \in \mathrm{Perm}(n)} a_{\sigma(1),1} \, a_{\sigma(2),2} \cdots a_{\sigma(n),n} \operatorname{sign}(\sigma)$

defined this way, we have all the desired properties!

$$D(v_2, v_1, v_3) = \sum_{\sigma \in \mathrm{Perm}(3)} a_{\sigma(1),1}\, a_{\sigma(2),2}\, a_{\sigma(3),3}\, \mathrm{sign}(\sigma)$$

1. If τ is a transposition

$$D(\overset{w_1}{v_{\tau(1)}}, \overset{w_2}{v_{\tau(2)}}, \dots, \overset{w_n}{v_{\tau(n)}})$$

$$B = \begin{array}{c} \begin{pmatrix} v_{\tau(1)} & v_{\tau(2)} & \cdots & v_{\tau(n)} \end{pmatrix} \\ \begin{matrix} w_1 & w_2 & & w_3 \end{matrix} \end{array}$$

$$= \sum_{\sigma \in \mathrm{Perm}(n)} b_{\sigma(1),1}\, b_{\sigma(2),2} \cdots b_{\sigma(n),n}\, \mathrm{sign}(\sigma)$$

$$= \sum_{\sigma \in \mathrm{Perm}(n)} a_{\sigma\tau(1),1} \cdots a_{\sigma\tau(n),n}\, \mathrm{sign}(\sigma)$$

$$= \sum_{\sigma \in \mathrm{Perm}(n)} a_{\sigma(1),1} \cdots a_{\sigma(n),n}\, \underbrace{\mathrm{sign}(\sigma\tau)}_{=(-1)\mathrm{sign}(\sigma)}$$

$\{\sigma : \sigma \in \mathrm{Perm}(n)\}$
$\{\sigma\tau : \sigma \in \mathrm{Perm}(n)\}$ $\rightsquigarrow$ show these two sets are the same

2. Linearity in each col

$$\begin{array}{c} \text{1st col} \\ \begin{pmatrix} a_{1,1} + u_1 \\ a_{2,1} + u_2 \\ \vdots \\ a_{n,1} + u_n \end{pmatrix} \end{array}$$

$D(v_1 + u, v_2, \dots, v_n)$

$$= \sum_{\sigma \in \text{Perm}(n)} (a_{\sigma(1),1} + u_{\sigma(1)}) [a_{\sigma(2),2} \cdots a_{\sigma(n),n}] \operatorname{sign}(\sigma)$$

$$= \sum_{\sigma} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n} \operatorname{sign} \sigma$$

$$+ \sum_{\sigma} u_{\sigma(1)} a_{\sigma(2),2} \cdots a_{\sigma(n),n} \operatorname{sign} \sigma$$

$$= D(v_1, \dots, v_n) + D(u, v_2, \dots, v_n)$$

3. $\det I = 1$?

$I = (a_{i,j})_{i,j=1}^{n}$

$a_{i,j} = 1$ if $i = j$

$a_{i,j} = 0$ if $i \neq j$

$a_{\sigma(i),i} = 0$ if $\sigma(i) \neq i$

$a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$

if even one $\sigma(i) \neq i$, then whole term is 0, so only surviving term is identity

$$\det I = \sum_{\sigma} \text{———} = \sum_{\sigma = id} \text{———}$$

$$= a_{1,1} a_{2,2} \cdots a_{n,n} \operatorname{sign}(id) = 1.$$

Ex.:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} E_1 A = \begin{pmatrix} 1 & 2 & -5 \\ 0 & 1 & 1 \\ 6 & -4 & 3 \end{pmatrix}$$

$\det E_1 = -1$

$$\xrightarrow[-6R_1]{} \begin{pmatrix} 1 & 2 & -5 \\ 0 & 1 & 1 \\ 0 & -16 & 33 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 49 \end{pmatrix}$$

$\det E_2 = 1$ $\det E_3 = 1$

$E_3 E_2 E_1 A$

upper triangular

$$\det(E_3 E_2 E_1 A) = 49$$

$$\det(E_3)\det(E_2)\det(E_1)\det(A) = 49$$

$$\det(A) = \frac{49}{(-1)(1)(1)} = -49.$$

Cofactor Expansion

$A = (a_{i,j})_{i,j=1}^{n}$. Let $A_{j,k}$ be the $(n-1)\times(n-1)$ matrix obtained by crossing out the j-th row and k-th col.

Thm 5.1 (cofactor expansion)

expand by row $\det A = \sum_{k=1}^{n} a_{j,k} (-1)^{j+k} \det(A_{j,k})$, j fixed

similarly,

expand by col $\det A = \sum_{i=1}^{n} a_{j,k} (-1)^{j+k} \det(A_{j,k})$, k fixed

Aside: Suppose $E_{n\times n}$ = row op matrix that doesn't change first row or use first row

Ex:

$$\det\begin{array}{c}\begin{array}{ccc} + & - & \cdot \end{array}\\ \begin{pmatrix}0&1&1\\1&2&-5\\6&-4&3\end{pmatrix}\end{array} = 0\begin{vmatrix}2&-5\\-4&3\end{vmatrix} - 1\begin{vmatrix}1&-5\\6&3\end{vmatrix}$$

$$+ 1\begin{vmatrix}1&2\\6&-4\end{vmatrix} = -1(3+30)$$

$$+1(-16)$$

$$= -49.$$

<u>Def.:</u> The numbers

$$C_{j,k} = (-1)^{j+k} \det A_{j,k}$$

are called <u>cofactors</u>

<u>Rmk:</u> Cofactor expansion is $O(n!)$

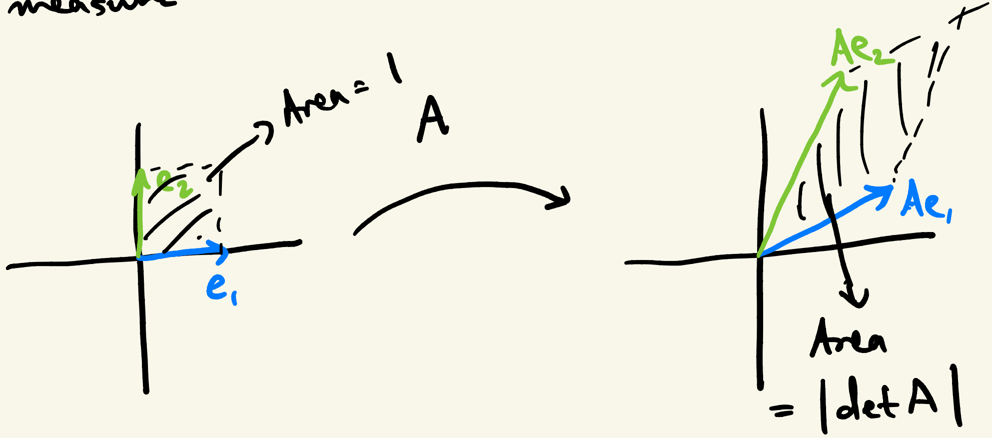
Row reduction is $O(n^3)$

The matrix $C = (C_{j,k})_{j,k=1}^{n}$ is called the <u>cofactor matrix</u>

<u>Thm 5.2</u> Let A inv, C its cofactor matrix

$$A^{-1} = \frac{1}{\det A} C^T$$

Rmk: $\det A$ can be thought of as a measure of volume distortion

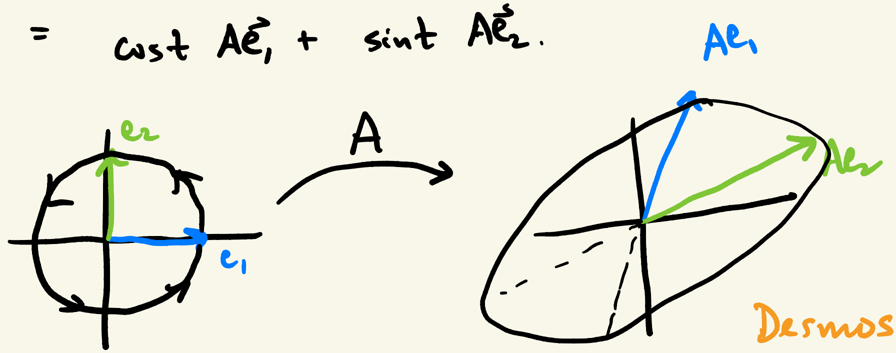


The sign of $\det A$ is a measure of whether orientation is preserved.

$$\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = (\cos t)\vec{e}_1 + (\sin t)\vec{e}_2$$

$$A\vec{x}(t) = A\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = A\begin{pmatrix} \cos t \\ 0 \end{pmatrix} + A\begin{pmatrix} 0 \\ \sin t \end{pmatrix}$$

$$= \cos t \, A\vec{e}_1 + \sin t \, A\vec{e}_2.$$



$\det A > 0 \implies$ CCW goes to CCW

$\det A < 0 \implies$ CCW goes to CW