Eigenvalues, eigenvectors

Let $A: V \to V$ linear.

A scalar λ is called an <u>eigenvalue</u> if there is some $\vec{v} \neq \vec{0}$ s.t.

$$A\vec{v} = \lambda \vec{v}.$$

$\vec{v}$ is called an <u>eigenvector</u>.

"eigen" = "self" or "own"

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} = \lambda I \vec{v}$$

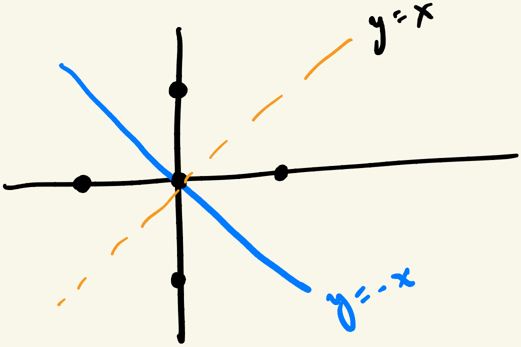
$$\iff (A - \lambda I)\vec{v} = \vec{0}.$$

<u>Def:</u> λ = eigval. $\ker(A - \lambda I)$ is called the <u>eigenspace</u> corresponding to λ.

The set of all eigenvalues of A is called the <u>spectrum</u> of A, denoted $\sigma(A)$.

Ex: Reflection about $y=-x$

$$A = \begin{array}{cc} Ae_1 & Ae_2 \\ \end{array}$$
$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$



what vectors are sent to a multiple of themselves? $Av = \lambda v \longrightarrow$ eigenvectors are ones s.t. v, Av, and 0 are collinear.

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector w/ eigenvalue -1

$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector w/ eigenvalue $+1$

Finding eigenvals

λ is an eigval if and only if

$\ker(A-\lambda I)$ is nontrivial (i.e.

$(A-\lambda I)\vec{x} = \vec{0}$ has a nonzero sol'n)

Since A is square, $A-\lambda I$ has nontrivial kernel if and only if $A-\lambda I$ is not invertible.

$\Longleftrightarrow \det(A-\lambda I) = 0.$

$$\boxed{\lambda \text{ is an eigval} \Longleftrightarrow \det(A-\lambda I) = 0}$$

If $A = n\times n$, then $\det(A-\lambda I)$ is a polynomial of degree n in the variable λ, called the <u>characteristic polynomial</u>.

<u>e.g.</u> $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$$A-\lambda I = \begin{pmatrix} -\lambda & -1 \\ -1 & -\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = \lambda^2 - 1$$

$$\lambda^2 - 1 = 0 \Longleftrightarrow \lambda = +1 \text{ or } \lambda = -1.$$

So we can find the spectrum of a matrix. what about a general linear map $T: V \to V$?

Take an arbitrary basis $\mathcal{B}$, and compute the matrix $[T]_{\mathcal{B}\mathcal{B}}$, and find its eigenvalues.

Does this depend on the choice of $\mathcal{B}$? No!

Def: Two $n \times n$ matrices A and B are <u>similar</u> if $A = SBS^{-1}$ for some invertible matrix S.

Note: $\det(A) = \det(S)\det(B)\det(S^{-1})$

$= \det(B)$, since $\det(S^{-1}) = \frac{1}{\det(S)}$

$$A - \lambda I = SBS^{-1} - \lambda SIS^{-1}$$

$$= S(BS^{-1} - \lambda I S^{-1}) = S(B - \lambda I)S^{-1}$$

$\Rightarrow \det(A - \lambda I) = \det(B - \lambda I)$.

$\Rightarrow$ similar matrices have the same characteristic polys $\to$ same eig. vals

$T: V \to V$. $\mathcal{A}, \mathcal{B}$ bases for V

$$[T]_{\mathcal{A}\mathcal{A}} = [I]_{\mathcal{A}\mathcal{B}} [T]_{\mathcal{B}\mathcal{B}} [I]_{\mathcal{B}\mathcal{A}}$$

$$[I]_{\mathcal{A}\mathcal{B}} = [I]_{\mathcal{B}\mathcal{A}}^{-1}$$

so $[T]_{\mathcal{A}\mathcal{A}}$ and $[T]_{\mathcal{B}\mathcal{B}}$ are similar

<u>Rmk:</u> eigenvalues need not be in $\mathbb{R}$

ex. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow (1-\lambda)^2 + 1$

$\lambda^2 - 2\lambda + 2$

<u>multiplicity</u>: If p is a polynomial, and λ is a root (i.e. $p(\lambda) = 0$), then

$p(z) = (z-\lambda) q(z)$. q polynomial

If $q(\lambda) = 0$, then q has a factor of $(z-\lambda)$, so $(z-\lambda)^2$ divides p, and so on

The largest pos. int. k s.t. $(z-\lambda)^k$ divides p is called the <u>multiplicity</u> of the root λ.

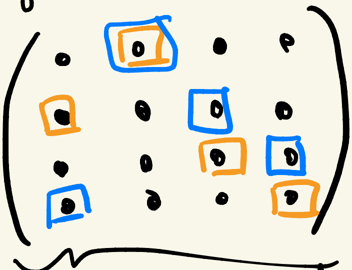
If λ is an eigenvalue, it is a root of $p(z) = \det(A - \lambda I)$. The multiplicity of this root is called the <u>multiplicity of the eig. val λ.</u>

Any $p(z)$ of degree n has exactly n complex roots (counting multiplicity).

$$p(z) = a_n (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)$$

$\lambda_i \in \mathbb{C}\ \forall i$, some λ_i's may be the same.

<u>Rule:</u> $\det A = \sum_{\sigma \in \text{sign}\,\sigma} a_{\sigma(1),1}\, a_{\sigma(2),2} \cdots a_{\sigma(n),n}\, \text{sign}(\sigma)$



each term involves selecting n entries, one from each col, can't share rows

The eigenvals of a triangular matrix are the diagonal entries:

$$A - \lambda I = \begin{pmatrix} a_{1,1}-\lambda & & \sim \\ & a_{2,2}-\lambda & \sim \\ & & \ddots & \sim \\ 0 & & & a_{n,n}-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (a_{1,1}-\lambda)(a_{2,2}-\lambda)\cdots(a_{n,n}-\lambda)$$

roots are exactly $a_{i,i}$, $i=1,\dots,n$

Diagonalization

$T: V \to V$. Is there a basis $\mathcal{B}$ for which $[T]_{\mathcal{B}\mathcal{B}}$ is diagonal?

Not always.

Suppose $A: V \to V$ has a basis consisting of eigenvectors $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$

$$A\vec{b}_1 = \lambda_1 \vec{b}_1, \ \ldots, \ A\vec{b}_n = \lambda_n \vec{b}_n$$

$$[A]_{\mathcal{B}\mathcal{B}} = \begin{array}{c} \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & \cdots & b_n \end{array} \\ \begin{pmatrix} \lambda_1 & 0 & \cdots & \vdots \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \end{array}$$

Conversely, if there is a basis for which $[A]_{\mathcal{B}\mathcal{B}}$ is diagonal, then those basis vectors are each eigenvectors.

Thm 2.1 : Let $A \in M_{n\times n}^{\mathbb{F}}$.

Then A can be written $A = SDS^{-1}$, D diagonal, S inv, if and only if there is a basis of $\mathbb{F}^n$ consisting of eigenvectors.

In this case, the diagonal entries of D are the eigenvals and the cols of S are the corresponding eigenvectors.

Pf: ($\Rightarrow$) Let $A = SDS^{-1}$, $D = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$, let $\vec{b}_1, \dots, \vec{b}_n$ be the cols of S = inv (in particular, this means $\vec{b}_1, \dots, \vec{b}_n$ is a basis).

$S = [I]_{\text{standard}, \mathcal{B}}$. $A = SDS^{-1} \Rightarrow D = S^{-1}AS$

$D = S^{-1}AS = [I]_{\mathcal{B}, st}\, A\, [I]_{st, \mathcal{B}} = [A]_{\mathcal{B}\mathcal{B}}$

this is to say the cols of S are eigenvectors!

($\Leftarrow$) follows from above. $\square$

Diagonalizing is useful for computing powers of maps/matrices

If $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $D^N = \begin{pmatrix} \lambda_1^N & & 0 \\ & \ddots & \\ 0 & & \lambda_n^N \end{pmatrix}$

$$A = SDS^{-1} \Rightarrow A^N = \overbrace{(SDS^{-1})(SDS^{-1})\cdots(SDS^{-1})}^{N \text{ times}}$$
$$= SD^NS^{-1}.$$

Thm 2.2: Let $\lambda_1, \ldots, \lambda_r$ be distinct eigenvals.

Let $\vec{v}_1, \ldots, \vec{v}_r$ be corresponding eigenvectors.

Then $\vec{v}_1, \ldots, \vec{v}_r$ are linearly independent.

Aside on Induction:

Suppose $S \subseteq \mathbb{N} = \{0, 1, 2, \ldots\}$ s.t.

$0 \in S$, and suppose $n \in S$ implies $n+1 \in S$.

Then $S = \mathbb{N}$.

$P(n)$: statement involving the number n.

e.g. $P(n) = 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Let $S = \{k \in \mathbb{N} : P(k) \text{ is true}\}$

If we show, $0 \in S$, then show

$n \in S \Rightarrow n+1 \in S$. Then $S = \mathbb{N}$

(i.e. $P(n)$ is true for all $n \in \mathbb{N}$).

In our example, $P(0)$: $0 = \frac{0(1)}{2}$, so $0 \in S$.

Suppose $P(n)$ is true for some n.

$$1 + \dots + n = \frac{n(n+1)}{2}.$$

Then $[1 + \dots + n] + n+1$

$$= \left[\frac{n(n+1)}{2}\right] + n+1 = \frac{n(n+1)}{2} + \frac{2n+2}{2}$$

$$= \frac{n^2 + 3n + 2}{2} = \frac{(n+2)(n+1)}{2}.$$

So $P(n+1)$ is true, provided $P(n)$ holds.

So $0 \in S$, $n \in S \Rightarrow n+1 \in S \Rightarrow S = \mathbb{N}$.

Pf of 2.2: If $r=1$, then the statement holds.

Suppose the theorem statement is true for $r-1$.

Let $\lambda_1, \ldots, \lambda_r$ be distinct eigenvals with $\vec{v}_1, \ldots, \vec{v}_r$ the corresponding eigenvectors.

$$c_1\vec{v}_1 + \cdots + c_r\vec{v}_r = 0 \quad (*)$$

Apply $A-\lambda_r I$ to both sides

$$c_1(A-\lambda_r I)\vec{v}_1 + \cdots + c_{r-1}(A-\lambda_r I)\vec{v}_{r-1} = \vec{0}$$

$$\Longrightarrow c_1\underbrace{(\lambda_1-\lambda_r)}_{\neq 0}\vec{v}_1 + \cdots + c_{r-1}\underbrace{(\lambda_{r-1}-\lambda_r)}_{\neq 0}\vec{v}_{r-1} = \vec{0}$$

$$\Longrightarrow c_1 = \cdots = c_{r-1} = 0.$$

so $(*)$ becomes $c_r\vec{v}_r = \vec{0} \Longrightarrow c_r = 0$ since $\vec{v}_r \neq \vec{0}$.

$\square$

Corollary: If $A: V \to V$ has $\underset{\dim V}{n}$ distinct eigvals, then it is diagonalizable.

Pf: A would then have n linearly indp eigenvectors. An LI list of length $\dim V$ is a basis. $\square$

Bases of subspaces (Direct sums)

Let $V_1, V_2, \ldots, V_p$ be <u>subspaces</u> of V.
We say $V_1, \ldots, V_p$ is a <u>basis</u> of V if every $\vec{v} \in V$ can be written <u>uniquely</u> as

$$\vec{v} = \vec{v}_1 + \cdots + \vec{v}_p, \quad \vec{v}_i \in V_i \quad i = 1, \ldots, p.$$

<u>Def:</u> $V_1, \ldots, V_p$ is <u>linearly indp</u> if

$$\vec{v}_1 + \cdots + \vec{v}_p = \vec{0}, \quad \text{each } \vec{v}_i \in V$$

implies $\vec{v}_i = \vec{0}$ $\forall i = 1, \ldots, p$.

<u>Def:</u> $V_1, \ldots, V_p$ is <u>spanning</u> if each $\vec{v} \in V$ can be expressed (not necessarily uniquely)

$$\vec{v} = \vec{v}_1 + \cdots + \vec{v}_p, \quad \vec{v}_i \in V_i, \quad i = 1, \ldots, p.$$

Rmk: The system of eigenspaces E_k of A

$$E_k := \ker(A - \lambda_k I)$$

is lin indp.

Rmk: A system of subspaces $V_1, \dots, V_p$ is a basis if and only if it is generating and lin indp.

Ex: Suppose $\vec{v}_1, \dots, \vec{v}_n$ is a basis for V.

Split the indices $1, 2, \dots, n$ into p subsets $\Lambda_1, \dots, \Lambda_p$. Let

$V_k = \operatorname{span}\{v_j : j \in \Lambda_k\}$. Then

$V_1, \dots, V_p$ is a basis of subspaces for V.

$$\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6$$

$\Lambda_1 = \{1\}$ $\Lambda_2 = \{2, 5\}$ $\Lambda_3 = \{3, 4, 6\}$

Prop: A system of eigenspaces is lin indp corresponding to diff eigvals

Induction on # of eigenspaces:

$$\underbrace{\begin{matrix} E_1, \ldots, & E_r, & E_{r+1} \\ \lambda_1 & \lambda_r & \lambda_{r+1} \end{matrix}}_{\text{distinct}}$$

Suppose $\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_r + \vec{v}_{r+1} = \vec{0}$ $\vec{v}_i \in E_i$

$$(A - \lambda_{r+1} I)(\vec{v}_1 + \cdots + \vec{v}_r + \vec{v}_{r+1}) = (A - \lambda_{r+1} I)\vec{0}$$

$$(A - \lambda_{r+1} I)\vec{v}_1 + \cdots + (A - \lambda_{r+1} I)\vec{v}_r + (A - \lambda_{r+1} I)\vec{v}_{r+1} = \vec{0}$$

$$(\lambda_1 - \lambda_{r+1})\vec{v}_1 + \cdots + (\lambda_r - \lambda_{r+1})\vec{v}_r + \cancel{(\lambda_{r+1} - \lambda_{r+1})\vec{v}_{r+1}} = \vec{0}$$

if any $\vec{v}_i$'s are $\neq \vec{0}$, then there is some linear dependence among $\vec{v}_i$'s.

So each $\vec{v}_i$ must be $\vec{0}$.

$\Rightarrow \vec{v}_{r+1} = \vec{0}$. $\square$

<u>Thm 2.6</u> Let $V_1, \ldots, V_p$ be a basis of subspaces. Let each V_k have a basis $\mathcal{B}_k$. Then $\bigcup_k \mathcal{B}_k$ is a basis for V.

<u>Lemma:</u> Let $V_1, \ldots, V_p$ be a lin indp list of <u>subspaces</u>, and suppose $\mathcal{B}_k$ is a lin indp list of <u>vectors</u> in V_k for each k. Then $\bigcup_k \mathcal{B}_k$ is a lin indp list in V. <u>of vectors</u>

<u>Pf:</u> Let n be the # of vectors in $\bigcup_k \mathcal{B}_k =: \mathcal{B}$. Order the set $\mathcal{B}$ as follows: first list all vectors in $\mathcal{B}_1$, then in $\mathcal{B}_2$, ..., then $\mathcal{B}_p$

$\mathcal{B}$ is a list of n vectors $\vec{b}_1, \ldots, \vec{b}_n$. split indices into sets $\Lambda_1, \ldots, \Lambda_p$ so that

$$\mathcal{B}_k = \{\vec{b}_j : j \in \Lambda_k\}$$

Suppose $c_1\vec{b}_1 + \ldots + c_n\vec{b}_n = \vec{0}$ (1).

write $\vec{v}_k = \sum_{j \in \Lambda_k} c_j \vec{b}_j$

Then (1) becomes

$$\vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_p = \vec{0}.$$

$V_1, \dots, V_p$ lin indp as subspaces

$$\Longrightarrow \vec{v}_k = \vec{0}, \quad k = 1, \dots, p.$$

i.e. $$\sum_{j \in \Lambda_k} c_j \vec{b}_j = \vec{0}$$

$\{b_j : j \in \Lambda_k\}$ are lin indp as vectors

$$\Longrightarrow c_j = 0 \quad \forall j \in \Lambda_k.$$

$$\Longrightarrow c_1 = c_2 = \dots = c_n = 0. \quad \square$$

<u>Pf of thm</u>: only remains to show spanning part. If $V_1, \dots, V_p$ is a basis of subspaces, then each $\vec{v} \in V$ can be written

$$\vec{v} = \vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_p, \quad \vec{v}_k \in V_k \quad k = 1, \dots, p.$$

Since $\{\vec{b}_j : j \in \Lambda_k\}$ is a basis for V_k,

$$v_k = \sum_{j \in \Lambda_k} c_j \vec{b}_j \text{ for some } c_j\text{'s}.$$

$$\Longrightarrow v = \sum_{j=1}^{n} c_j \vec{b}_j. \quad \square$$

Criterion for diagonalizability

$D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ has n eigenvals (counting multiplicity). So if $A: V \to V$ is diagonalizable, it must have n eigenvalues (counting multiplicity).

<u>Thm 2.8</u>: Suppose $A: V \to V$ linear has $n = \dim V$ eigenvalues (if $\mathbb{F} = \mathbb{C}$, this assumption is redundant). Then A is diagonalizable iff for each eigenvalue λ, the dimension of $\ker(A - \lambda I)$ coincides with the multiplicity of λ.

<u>Pf</u>: ($\Longrightarrow$) If A has a diagonal repr $[A]_{\mathcal{B},\mathcal{B}}$, then each eigenvalue has a corresponding eigenvector.

e.g. if diagonal contains same eigenvalue λ j times, there will be j lin indp eigenvectors associated to λ.

$$[A]_{\mathcal{B},\mathcal{B}} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Suppose $A\vec{v} = \lambda\vec{v}$

Then $(A - \lambda I)\vec{v} = 0$

$$[A - \lambda I]_{\mathcal{B},\mathcal{B}} [v]_{\mathcal{B}}$$

$$= \begin{pmatrix} 0 & \lambda_2 & 0 & \lambda_4 & & \\ & & & & \ddots & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{B}}$$

$\Leftrightarrow$ dimension of nullspace is exactly multiplicity of λ.

($\Leftarrow$) Let $\lambda_1, \ldots, \lambda_p$ be the distinct eigenvalues of A, $E_k = \ker(A - \lambda I)$ the corresponding eigenspaces. $E_1, \ldots, E_p$ are lin indp as subspaces. Let $\mathcal{B}_k$ be a basis of E_k.

$\mathcal{B} = \bigcup_k \mathcal{B}_k$ is a lin indp list of vectors in V.

By assumption, $\underset{\#\text{ of vectors in } \mathcal{B}_k}{\dim E_k} =$ multiplicity of λ_k.

Sum of multiplicities $= n$, so $\mathcal{B}$ has $n = \dim V$ vectors in it. It's a basis (of eigenvectors), so A is diagonalizable. $\square$

Ex.: $A = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}$

$$\det \begin{vmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 16$$

$$= \lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3)$$

Two eigvals $\longrightarrow$ $\lambda_1 = 5, \ \lambda_2 = -3$

$$A - 5I = \begin{pmatrix} 1-5 & 2 \\ 8 & 1-5 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 8 & -4 \end{pmatrix}$$

ker$(A - 5I)$: $\begin{pmatrix} -4 & 2 \\ 8 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} \boxed{2} & -1 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{2}x_2 \\ x_2 \end{pmatrix}, x_2 \in \mathbb{R} \longrightarrow$ ker is spanned by $\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\text{eigenvector}}$

$$A + 3I = \begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}$$

<u>Ex:</u> A non-diagonalizable matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$p(\lambda) = (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$ (multiplicity)

$\dim \ker (A - 1I) = 1$

so $\dim$ (eigenspace for $\lambda = 1$) $\neq$ multiplicity of $\lambda = 1$

Ex: (Complex eigenvalues)

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 2^2 = 0$$

$$\Rightarrow \lambda = 1 \pm 2i$$

$$A - (1+2i) = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}$$

$\longrightarrow \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigvector

If $Av = \lambda v$, then $\overline{Av} = \overline{\lambda v} \rightarrow \bar{A}\bar{v} = \bar{\lambda}\bar{v}$

$A\bar{v} = \bar{\lambda}\bar{v}$.

So for $\lambda = 1-2i$, $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eig. vec.

$$A = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}$$

Block matrices

Can slice a matrix into blocks

$$\left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & \cdot \\ \cdot & A \quad \cdot & \cdot & B \quad \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & C \quad \cdot & \cdot & D \quad \cdot \end{array}\right) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,q} \\ A_{2,1} & \cdots & A_{2,q} \\ \vdots & \ddots & \vdots \\ A_{p,1} & \cdots & A_{p,q} \end{pmatrix}$$

All blocks in a row have same # of rows

All blocks in a col have same # of cols.

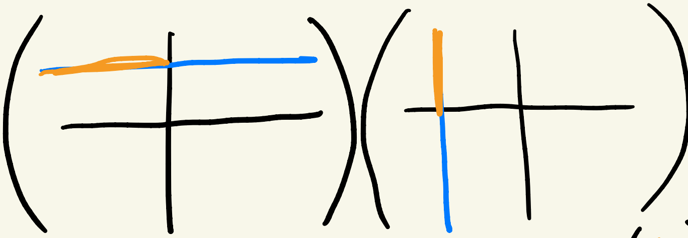
Let A be square $(k \times k)$

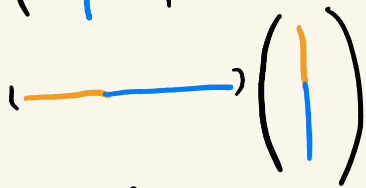
$$\det \begin{pmatrix} I_n & * \\ 0 & A \end{pmatrix} = \det \left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & * \\ 0 & & 1 & \\ \hline & 0 & & A \end{array}\right)$$

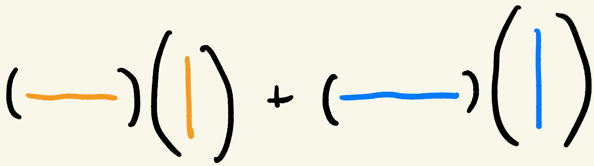
cofactor expansion along 1st col

$$\det \begin{pmatrix} I_n & * \\ 0 & A \end{pmatrix} = \underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{n} \det A.$$

Block matrices can be multiplied like ordinary matrices, provided dimensions match

$a_{ij} =$ 

The observation is that 

$=$ 

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & C \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

$$\Rightarrow \det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A \det C$$

$$\det\left(\begin{pmatrix} A & * \\ 0 & B \end{pmatrix} - \lambda I_n\right) \qquad A, B \text{ square}$$

$$= \det\left(A - \lambda I_l\right) \det\left(B - \lambda I_k\right) \qquad k+l=n$$

Now suppose $\overbrace{\vec{v}_1, \ldots, \vec{v}_n}^{\mathcal{B}}$ is a basis of V, and that the first k vectors $\vec{v}_1, \ldots, \vec{v}_k$ are eigenvectors of $A: V \to V$ corresponding to an eigenvalue λ^*.

(i.e. $\vec{v}_1, \ldots, \vec{v}_k \in \ker(A - \lambda^* I)$). Then

$$[A]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} \lambda^* I_k & * \\ 0 & B \end{pmatrix}$$

$B \in M_{n-k,\, n-k}$

The characteristic poly of A has the form

$$p(\lambda) = (\lambda^* - \lambda)^k q(\lambda), \quad q = \text{polynomial}$$

$\Rightarrow$ the multiplicity of λ^* is at least as large as $\dim \ker(A - \lambda^* I)$.

$\Rightarrow \dim \ker(A - \lambda^* I) \leq$ multiplicity of λ^*.

Ex: Find all square roots of

$$A = \begin{pmatrix} 5 & 2 \\ -3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

$$\det A - \lambda I = \det \begin{pmatrix} 5-\lambda & 2 \\ -3 & -\lambda \end{pmatrix} = -(5-\lambda)\lambda + 6$$

$$= \lambda^2 - 5\lambda + 6$$

$$= (\lambda-3)(\lambda-2)$$

$\underline{\lambda = 3}$ $\quad A - 3I = \begin{pmatrix} 2 & 2 \\ -3 & -3 \end{pmatrix} \longrightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underline{\lambda = 2:}$

$A - 2I = \begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \qquad \begin{matrix} -3+2 \\ = -1 \end{matrix}$

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} \pm\sqrt{3} & 0 \\ 0 & \pm\sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$

four possible square roots

$$B = \begin{pmatrix} \pm\sqrt{3} & \pm 2\sqrt{2} \\ \mp\sqrt{3} & \mp 3\sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mp 3\sqrt{3} \pm 2\sqrt{2} & \mp 2\sqrt{3} \pm 2\sqrt{2} \\ \pm 3\sqrt{3} \mp 3\sqrt{2} & \pm 2\sqrt{3} \mp 3\sqrt{2} \end{pmatrix}$$