Invertibility and Equations

<u>Thm:</u> Let $A : X \to Y$ be linear. Then A is invertible if and only if for any $\vec{b} \in Y$ the eqn
$$A\vec{x} = \vec{b}$$
has a unique solution $\vec{x} \in X$.

<u>Pf:</u> $(\Rightarrow)$ Suppose A invertible. $\vec{x} = A^{-1}\vec{b}$ solves $A\vec{x} = \vec{b}$. Suppose $\vec{x}_1$ also solves the eqn ($A\vec{x}_1 = \vec{b}$).

Then
$$A^{-1}A\,\vec{x}_1 = A^{-1}\vec{b}$$
$$\Rightarrow \vec{x}_1 = \vec{x}.$$

$(\Leftarrow)$ Suppose $A\vec{x} = \vec{y}$ has a unique soln $\forall \vec{y} \in Y$. Call the unique sol'n associated to $\vec{y}$ $B(\vec{y}) \in X$.
Then B is a function $B : Y \to X$. We check that B is linear. Let $\vec{y}_1, \vec{y}_2 \in Y$, $\alpha, \beta \in \mathbb{F}$.
Let $\vec{x}_1 = B(\vec{y}_1)$, $\vec{x}_2 = B(\vec{y}_2)$.

$$A(\alpha\vec{x}_1 + \beta\vec{x}_2) = \alpha A\vec{x}_1 + \beta A\vec{x}_2$$

$$= \alpha\vec{y}_1 + \beta\vec{y}_2.$$

so $\alpha\vec{x}_1 + \beta\vec{x}_2$ is the unique solution $B(\alpha\vec{y}_1 + \beta\vec{y}_2)$, i.e. B is linear.

Now, let $\vec{x} \in X$, $\vec{y} = A\vec{x}$. By def'n,

$\vec{x} = B\vec{y}$. So

$$BA\vec{x} = B(\vec{y}) = \vec{x}.$$

$\Rightarrow BA = I_X$.

Now let $\vec{y} \in Y$, set $\vec{x} = B\vec{y}$, so that $\vec{y} = A\vec{x}$. Then

$$AB\vec{y} = A\vec{x} = y.$$

$\Rightarrow AB = I_Y$. $\square$

<u>Corollary:</u> An $m \times n$ matrix is invertible if and only if its columns form a basis for $\mathbb{F}^m$ ($\mathbb{R}^m$ or $\mathbb{C}^m$)

Subspaces

A **subspace** $V_0 \subset V$ of V is a nonempty subset of V s.t.

(1) $\alpha\vec{u} + \beta\vec{v} \in V_0 \quad \forall \vec{u}, \vec{v} \in V_0, \ \alpha, \beta \in \mathbb{F}$

i.e. A subspace is a subset of V that is also a vector space w/ the same addition and scaling.

Condition (1) ensures this.

examples:

1) the trivial subspaces
   $\{0\}$, V.
   $\emptyset$ is **not** a subspace

2) $A: V \to W$ linear
   the **nullspace** (or **kernel**) of A defined by
    Null A Ker A

$$\text{Ker } A = \{\vec{v} \in V : A\vec{v} = \vec{0}\}$$

why is this a subspace?

3) $A: V \to W$ linear

The <u>range</u> (or <u>image</u> of A)

$$\operatorname{Ran} A = \{\vec{w} \in W : \vec{w} = A\vec{v} \text{ for some } \vec{v} \in V\}$$

4) Given a list of vectors $\vec{v}_1, \dots, \vec{v}_r \in V$, its <u>span</u> $\mathcal{L}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r)$ is the set

$$\mathcal{L}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r) = \{\alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r : \alpha_i \in \mathbb{F} \ \forall i = 1, \dots, r\}$$

i.e. the collection of all possible linear combinations

If A is a matrix (i.e. $A: \mathbb{R}^m \to \mathbb{R}^n$), then $\operatorname{Ran} A$ is simply the span of the columns $\vec{a}_1, \dots, \vec{a}_m$.

Systems of Linear Eqns

m equations, n unknowns $x_1, \dots, x_n$

$$\begin{cases} a_{1,1}x_1 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + \cdots + a_{2,n}x_n = b_2 \\ \quad \vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n = b_m \end{cases} \qquad (*)$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

Then $(*)$ becomes $A\vec{x} = \vec{b}$.

Solving $(*)$ is then equivalent to finding all $\vec{x}$ s.t. $A\vec{x} = \vec{b}$.

Another way to write (*) is

$$x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{b}$$

where $\vec{a}_k$ is the k-th col of A.

$$a_k = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{m,k} \end{pmatrix}$$

Everything about the system is contained in the <u>augmented</u> matrix

$$\left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{array}\right)$$

<u>Row operations</u>

1) Row exchange: interchange two rows of the matrix
2) Scaling: scaling a row by a nonzero scalar
3) Row replacement: replace k-th row by its sum with a multiple of the j-th row for k, j of our choosing

Clear that 1) and 2) don't alter the set of solutions

For 3) : If we apply a type 3 operation, any $\vec{x}$ that satisfies the old system will satisfy the new system.

Operation 3 is reversible;

row k + α (row j) $\longrightarrow$ row k

can be reversed by

row k $-\alpha$ (row j) $\longrightarrow$ row k.

tells us $S_{old} \subseteq S_{new}$

using same argument on reverse op. , $S_{new} \subseteq S_{old}$.

$\Longrightarrow S_{new} = S_{old}$

If we use $A\vec{x} = \vec{b}$ form of the system, we can express these row ops as matrix multiplication

$$\begin{array}{c} \\ \\ j \rightarrow \\ \\ \\ k \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} 1 & & & & & & & & O \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & 1 & & \\ & & & \vdots & 1 & & \vdots & & \\ & & & \vdots & & \ddots & \vdots & & \\ & & & \vdots & & & 1 & & \\ & & & 1 & & & 0 & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ O & & & & & & & & & 1 \end{pmatrix} =: E_1$$

multiplication on left by this matrix exchanges row j and row k

$$E_2 := \begin{array}{c} \\ k \to \\ \\ \end{array} \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

scales k-th row by a

$$E_3 := \begin{array}{c} \\ j \\ k \\ \\ \end{array} \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & \vdots & \ddots & & \\ & a & \cdots & 1 & \\ 0 & & & & 1 \end{pmatrix}$$

adds $a(\text{row } j)$ to row k

$$E_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \vec{e}_1 + \vec{x}_2 \vec{e}_2 + \dots + x_j \vec{e}_k + x_{j+1} \vec{e}_{j+1} + \dots + x_k \vec{e}_j + \dots + x_n \vec{e}_n$$

same except now x_j and x_k are swapped.

Since E_1 does this to each column, $E_1 A$ exchanges rows j and k in A

$$E_3 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ a x_j + x_k \\ \vdots \\ x_n \end{pmatrix}$$

$$x_j \begin{pmatrix} 0 \\ 1 \\ \vdots \\ a \\ \vdots \\ 0 \end{pmatrix} \begin{array}{c} \\ j \\ \\ k \\ \\ \\ \end{array}$$

Each of these three matrices is invertible.

E_1 is its own inverse

E_2^{-1} is obtained by replacing a with $1/a$

E_3^{-1} is obtained by replacing a with $-a$

If E is one these special matrices

$$A\vec{x} = \vec{b} \iff EA\vec{x} = E\vec{b}$$

($\Rightarrow$) clear

($\Leftarrow$) multiply by E^{-1}.

So we see row ops don't change set of sol'ns.

Row reduction

The main step:

a) Find leftmost nonzero column

b) if necessary, apply row exchanges to make first entry of this col nonzero. This entry will be called a **pivot** (can also scale to make pivot $=1$)

c) "kill" (make $=0$) all nonzero entries below the pivot by adding an appropriate multiple of first row to rows $2, 3, \ldots, m$.

Apply main step to matrix A, then "forget" the first row; it is now "frozen". Apply main step to remaining rows.

Process terminates after $\leq m$ main steps.

E.g.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 3x_1 + 2x_2 + x_3 = 7 \\ 2x_1 + x_2 + 2x_3 = 1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 7 \\ 2 & 1 & 2 & 1 \end{array}\right)$$

$$\longrightarrow \begin{array}{c} \\ -3R_1 \\ -2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & 4 \\ 0 & -3 & -4 & -1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -3 & -4 & -1 \end{array}\right) \longrightarrow \begin{array}{c} \\ \\ +3R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -4 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

so $x_3 = -2$

$\Rightarrow x_2 + 2(-2) = -1 \Rightarrow x_2 = 3$

$\Rightarrow x_1 + 2 \cdot 3 + 3(-2) = 1 \Rightarrow x_1 = 1.$

$$\vec{x} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

Instead of back substitution, can do row reduction backwards to clear entries above diagonal

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right) -2R_2$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Def.: A matrix is in echelon form if it satisfies:

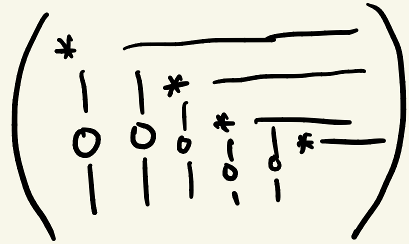
1) All zero rows are at the bottom

In a nonzero row, call the leftmost nonzero entry the **leading entry**.

2) For a nonzero row, the leading entry is strictly to the right of leading entry in row above.

The leading entry of each row (in echelon form) is called a **pivot**. These are exactly the same pivots as above.

why are they same?



row reduction yields echelon form

backward row reduction yields reduced echelon form (RREF)

3. All pivot entries are 1

4. All entries above a pivot are 0.

example of RREF

$$\left(\underbrace{\begin{matrix} \boxed{1} & 2 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 5 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}}_{\vec{b}} \right)$$

x_1, x_3, x_5 are pivot variables

solution is easy to read off once in RREF

$$\begin{cases} x_1 + 2x_2 = 1 \\ x_3 + 5x_4 = 2 \\ x_5 = 3 \end{cases}$$

move non-pivot variables to one side

$$\begin{cases} x_1 = 1 - 2x_2 \\ x_3 = 2 - 5x_4 \\ x_5 = 3 \end{cases}$$

$A\vec{x} = b$ has solution

$$\vec{x} = \begin{pmatrix} 1-2x_2 \\ x_2 \\ 2-5x_4 \\ x_4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix}$$

$x_2, x_4 \in \mathbb{R}$

x_2 and x_4 are <u>free variables</u>

any choice of them yields a valid sol'n, and any solution is obtained in this way.

<u>This always works</u>: any pivot entry is the only pivot in that row, so a pivot variable can always be written in terms of the free/nonpivot variables.

E.g.

$$\begin{cases} 2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 = 1 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 = 2 \\ 4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 = 6 \end{cases}$$

$$\downarrow$$

$$\left(\begin{array}{ccccc|c} 2 & -2 & -1 & 6 & -2 & 1 \\ 1 & -1 & 1 & 2 & -1 & 2 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 2 & -2 & -1 & 6 & -2 & 1 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{array}\right)$$

$$\downarrow$$

$$\begin{array}{c} \\ -2R_1 \\ -4R_1 \end{array} \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & -3 & 2 & 0 & -3 \\ 0 & 0 & 1 & -1 & 3 & -2 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & -1 & 3 & -2 \\ 0 & 0 & -3 & 2 & 0 & -3 \end{array}\right)$$

$$\begin{array}{r} \\ \\ +3R_2 \end{array}\left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & -1 & 3 & -2 \\ 0 & 0 & 0 & -1 & 9 & -9 \end{array}\right) \quad \text{REF}$$

$$\left(\begin{array}{ccccc|c} \boxed{1} & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & \boxed{1} & -1 & 3 & -2 \\ 0 & 0 & 0 & \boxed{1} & -9 & 9 \end{array}\right)$$

$$\begin{array}{r} -2R_3 \\ +R_3 \\ \\ \end{array}\left(\begin{array}{ccccc|c} \boxed{1} & -1 & 1 & 0 & 17 & -16 \\ 0 & 0 & \boxed{1} & 0 & -6 & 7 \\ 0 & 0 & 0 & \boxed{1} & -9 & 9 \end{array}\right)$$

$$\begin{array}{r} -R_2 \\ \\ \\ \end{array}\left(\begin{array}{ccccc|c} \boxed{1} & -1 & 0 & 0 & 23 & -23 \\ 0 & 0 & \boxed{1} & 0 & -6 & 7 \\ 0 & 0 & 0 & \boxed{1} & -9 & 9 \end{array}\right)$$

Analyzing the pivots

Def: $A\vec{x} = \vec{b}$ is <u>consistent</u> if it has a sol'n, <u>inconsistent</u> otherwise.

Prop: A system is inconsistent iff there is a pivot in the last col of the EF of the augmented matrix (i.e. EF has a row $(0 \ 0 \ \dots \ 0 \mid b)$, $b \neq 0$)

Pf: $\square$

Prop: Consider a system $A\vec{x} = \vec{b}$.

1) A sol'n (if it exists) is unique iff there are no free variables

2) $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ iff EF of A has a pivot in every row

3) $A\vec{x} = \vec{b}$ has a unique sol'n for each $\vec{b}$ iff EF of A has a pivot in every col and row.

Pf:

(1) is immediate, since no free vars means each variable has a fixed value $x_i = c_i$.

$$\vec{0}^T \longrightarrow \begin{pmatrix} * & & & \\ 0 & * & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & \end{pmatrix}$$

zero rows are "pushed down" by row reduction

(2) If A has a pivot in every row, then the augmented matrix will <u>not</u> have a pivot in last col, so sol'n exists

$$\left(\begin{array}{cccc|c} * & & & & \\ & * & & & \\ & & \vdots & & \\ & & & * & \end{array}\right)$$

<u>Converse:</u> Suppose the echelon form A_e of A has a zero row.

$$A_e = \underbrace{E_N \dots E_2 E_1}_{E} A$$

if A_e has a 0 row, then the last row must also be 0.

So set $b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Then $A_e \vec{x} = b$ is inconsistent

$$\Rightarrow \quad \underset{\displaystyle A\vec{x}}{\underset{\parallel}{E^{-1} A_e \vec{x}}} = E^{-1}\vec{b} \text{ is inconsistent}$$

why?

Aside: If $A\vec{x} = \vec{b}$, then $EA\vec{x} = E\vec{b}$

If E is invertible, then converse is true:

$EA\vec{x} = E\vec{b} \Rightarrow A\vec{x} = \vec{b}$.

(3) follows from (1) + (2). $\square$

Thm: The reduced echelon form of a matrix A is unique.

Pf: Suppose R and S are both RREF matrices of A and $R \neq S$. Find the first column (k) where R and S differ. Form the matrix R' by taking k-th col of R and every pivot col to the left of it. Form S' from S similarly.

For example if

$$R = \begin{pmatrix} \boxed{1} & 2 & 0 & 3 & 5 \\ 0 & 0 & \boxed{1} & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} \boxed{1} & 2 & 0 & 7 & 9 \\ 0 & 0 & \boxed{1} & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

then

$$R' = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}, \quad S' = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{pmatrix}.$$

In general,

$$R' = \left(\begin{array}{c|c} I_n & \vec{r}' \\ \hline O & \vec{0} \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} I_n & 0 \\ \hline O & \begin{matrix} 1 \\ 0 \\ \vdots \end{matrix} \end{array}\right)$$

if pivot

$$S' = \left(\begin{array}{c|c} I_n & \vec{s}' \\ \hline O & \vec{0} \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} I_n & 0 \\ \hline O & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right)$$

R' and S' are row equivalent, since deleting columns doesn't affect row equivalence.

Also, $R' \neq S'$.

View R' and S' as augmented matrices. R' and S' have the same set of solutions since they are row equivalent.

So either $\vec{r}' = s'$ or they're both inconsistent.

In either case, $R' = S'$, which is a contradiction.

$\square$

Rmk: In echelon form, any row or col has at most one pivot.

3.1 Corollaries about linear independence, bases

Prop: Let $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$, and let

$A = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m]$ be an $n \times m$ matrix

with cols $\vec{v}_i, i=1,\dots,m$. Then

(1) $\vec{v}_1, \dots, \vec{v}_m$ are lin indp iff echelon form of A has a pivot in every col.

(2) $\vec{v}_1, \dots, \vec{v}_m$ spans $\mathbb{R}^n$ iff echelon form has pivot in each row

(3) $\vec{v}_1, \dots, \vec{v}_m$ is a basis for $\mathbb{R}^n$ iff EF of A has a pivot in each col and each row.

Pf: $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ are LI iff

$$x_1 \vec{v}_1 + \dots + x_m \vec{v}_m = \vec{0}$$

has only one solution, $x_1 = x_2 = \dots = x_m = 0$. i.e.

$A\vec{x} = \vec{0}$ has the unique solution $\vec{x} = 0$.

This happens iff A has a pivot in every col. (1)

(2) $v_1, \dots, v_m$ spans $\mathbb{R}^n$ iff

$$x_1\vec{v}_1 + \dots + x_m\vec{v}_m = \vec{b}$$

has a sol'n for every $\vec{b} \in \mathbb{R}^n$.

This happens iff A has a pivot in every row.

(3) Combine (1) and (2). $\square$

<u>Prop</u>: Any lin indp list $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ cannot have more than n vectors in it.

<u>Pf</u>:

$$A = \left(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\right) \quad n \times m \text{ matrix}$$

Then A has a pivot in each col.

If $m > n$, this is not possible. (Pigeonhole Principle) $\square$

<u>Prop</u>: Any two bases of V have the same number of vectors in them.

<u>Pf</u>: Let $v_1, \dots, v_n$ and $w_1, \dots, w_m$ be bases of V. For the sake of concreteness, suppose $n \leq m$.

The map $A : \mathbb{F}^n \to V$ defined by

$$A(\vec{e}_k) = \vec{v}_k, \quad k = 1, \dots, n$$

is an isomorphism.

$A^{-1}: V \to \mathbb{F}^n$ is also an isomorphism, so

$A^{-1}\vec{w}_1, \ldots, A^{-1}\vec{w}_m$ is a basis.

So $m \le n$. $\Rightarrow m = n$. $\square$

<u>Def:</u> This number is called the <u>dimension</u> of V.

<u>Rmk:</u> Any basis of $\mathbb{R}^n$ has exactly n vectors in it.

<u>Prop:</u> Any spanning set in $\mathbb{R}^n$ must have at least n vectors.

<u>Pf:</u> Let $\vec{v}_1, \ldots, \vec{v}_m$ span $\mathbb{R}^n$. Then

$$A = \left(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\right)$$

has a pivot in every row (i.e. it has n pivots. $\Rightarrow n \le m$. $\square$

Revisit RREF uniqueness pf. <u>Pivot cols</u>

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

deleting cols can break echelon form,
but deleting last col is fine.

Prop 3.6: A matrix is invertible iff its echelon form has a pivot in every row and every col

Pf: we've seen $A\vec{x} = \vec{b}$ has a unique sol'n for each $\vec{b}$ iff echelon form of A has a pivot in each row and col.

OTOH A is invertible iff $A\vec{x} = \vec{b}$ has unique sol for each $\vec{b}$. $\square$

Cor: An invertible matrix must be square.

Prop 3.8 If A $(n \times n)$ is left inv, or right inv, then it is invertible.

Pf: A inv $\Longleftrightarrow$ $A\vec{x} = \vec{b}$ unique sol'n for each $\vec{b}$

$\Longleftrightarrow$ A has pivot in each row, each col

Suppose A left inv. Then $A\vec{x} = \vec{0}$ has $\vec{0}$ as its only solution ($BA\vec{x} = B\vec{0} \Rightarrow \vec{x} = \vec{0}$).

So no free vars $\Rightarrow$ each col has pivot.

$A = n \times n$, so each col has a pivot, so A is invertible.

Let $\vec{b} \in \mathbb{R}^n$. Suppose $AC = I$. Let $\vec{x} = C\vec{b}$.

$$A\vec{x} = AC\vec{b} = I\vec{b} = \vec{b}$$

So $A\vec{x} = \vec{b}$ always has sol'n $\vec{x} = C\vec{b}$.

So A has a pivot in every col. $\square$

Find A^{-1} by row reduction

<u>Def:</u> two matrices are <u>row equivalent</u> if row ops can transform one into the other.

(write $A \sim B$).

$$A = \underbrace{E_N E_{N-1} \cdots E_1}_{\text{row ops}} B.$$

<u>Obs:</u> Any invertible matrix A is row eq. to I_n.

<u>Algorithm to compute A^{-1} (if A is invertible)</u>

- form the $n \times 2n$ matrix

$$(A \mid I_n)$$

perform row ops to reduce A to I_n

$$\downarrow$$

$$(I_n \mid A^{-1})$$

<u>why?</u> $A\vec{x} = \vec{b}$ has solution $\vec{x} = A^{-1}b$ if A is inv.

Note $A^{-1}\vec{e}_k$ is the k-th col of A^{-1}.

So k-th col of A^{-1} is the solution to $A\vec{x} = \vec{e}_k$. The above algorithm then solves $A\vec{x} = \vec{e}_k$ for each $k = 1, \ldots, n$ simultaneously.

Another way:

Let $E = E_N \dots E_2 E_1$ be the row ops taking A to I

i.e. $EA = I_n$. Then $E = A^{-1}$, so that $EI_n = A^{-1}$.

So $E(A \mid I) = (I_n \mid A^{-1})$. $\square$

Thm: A invertible $\Longrightarrow A = E_N \dots E_2 E_1$, each E_i elementary.

Pf: $E = A^{-1}$, so $A = E^{-1}$

$A = E_1^{-1} E_2^{-1} \dots E_N^{-1}$. $\square$

Dimension, Finite dimensional space

Def: A vector space is <u>finite-dimensional</u> if it has a finite basis.

Prop: A vector space V is f.d. iff it has a finite spanning set.

Obs: If V has basis $\vec{v}_1, \dots, \vec{v}_n$, then

$$A : V \to \mathbb{R}^n \quad (\text{or } \mathbb{F}^n)$$

$$A\vec{v}_k = \vec{e}_k, \; k = 1, \dots, n$$

is an isomorphism.

Prop 5.2 Any LI list in a fdvs V has no more than $\dim V$ vectors in it.

Pf: Let $\vec{v}_1, \dots, \vec{v}_m$ be LI. Let $A : V \xrightarrow{\cong} \mathbb{R}^n$.

Then $A\vec{v}_1, \dots, A\vec{v}_m$ is LI in $\mathbb{R}^n \Rightarrow m \le \underset{\dim V}{\underset{\shortparallel}{n}}$. $\square$

Prop 5.3 Any spanning set in a fdvs V has at least $\dim V$ vectors in it.

Pf: Let $\vec{v}_1, \ldots, \vec{v}_m \in V$ span V. $A: V \to \mathbb{R}^n$ iso. Then $A\vec{v}_1, \ldots, A\vec{v}_m$ spans $\mathbb{R}^n$, so $m \geq n = \dim V$. $\square$

Completing an LI system to a basis

Prop 5.4 An LI list of vectors $\vec{v}_1, \ldots, \vec{v}_r \in V$ = fdvs can be completed to a basis.

Pf: Let $n = \dim V$. Take some $v_{r+1} \notin \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_r\}$. Then $v_1, \ldots, v_r, v_{r+1}$ is LI. (why?) If still not spanning, find some v_{r+2}, etc. Process will terminate since an LI list can't have more than $n = \dim V$ vectors. $\square$

Thm 5.5 Let $V \subseteq W$ be a subspace.
W = f.d.v.s. Then V is f.d. and $\dim V \leq \dim W$.
Moreover, if $\dim V = \dim W$, then $V = W$.

Pf: If $V = \{0\}$, done. ✓

Otherwise let $\vec{v}_1$ be a nonzero vector in V.
If this doesn't span V, find $v_2 \notin \mathrm{span}(v_1)$.

Continue process:
at each step, list is lin. indep.
Eventually must stop, since length of list $\leq \dim W$.
(can't keep finding vectors not in $\mathrm{span}(v_1, \ldots, v_r)$).

Result is $v_1, \ldots, v_m$ = spanning, LI. □

General sol'n of linear systems

<u>Def:</u> $A\vec{x} = \vec{b}$ is <u>homogeneous</u> if $\vec{b} = 0$.
i.e. the system is of the form $A\vec{x} = \vec{0}$.

For $A\vec{x} = \vec{b}$, the <u>associated homogeneous system</u> is $A\vec{x} = \vec{0}$.

<u>Thm 6.1</u> Suppose $\vec{x}_1$ satisfies $A\vec{x} = \vec{b}$ $(A : V \to W)$
(i.e. $A\vec{x}_1 = \vec{b}$). Let

$$H = \{ \vec{x} \in V : A\vec{x} = \vec{0} \}$$

Then the set

$$\vec{x}_1 + H := \{ \vec{x}_1 + \vec{x}_h : \vec{x}_h \in H \}$$

is the set of all sol'ns to $A\vec{x} = \vec{b}$.

(General sol'n of $A\vec{x} = \vec{b}$ = General sol'n of $A\vec{x} = \vec{0}$ + Particular sol'n of $A\vec{x} = \vec{b}$)

<u>Pf:</u> Suppose $\vec{x}_1$ is s.t. $A\vec{x}_1 = \vec{b}$. Suppose $\vec{x}_h$ is s.t. $A\vec{x}_h = \vec{0}$. Then

$$A(\vec{x}_1 + \vec{x}_h) = A\vec{x}_1 + A\vec{x}_h = A\vec{x}_1 = \vec{b}.$$

so $\vec{x}_1 + H \subseteq$ all sol'ns to $A\vec{x} = \vec{b}$.

Suppose $A\vec{x} = \vec{b}$. Set $\vec{x}_h := \vec{x} - \vec{x}_1$.

$$A\vec{x}_h = A(\vec{x} - \vec{x}_1) = A\vec{x} - A\vec{x}_1 = \vec{b} - \vec{b} = \vec{0}.$$

So $\vec{x}_h \in H$. So any sol'n of $A\vec{x} = \vec{b}$ can be written as $\vec{x} = \vec{x}_1 + \vec{x}_h$, w/ $\vec{x}_h \in H$. $\square$

Ex.:

$$\begin{array}{cc} A & & \vec{b} \\ \begin{pmatrix} 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{pmatrix} & \vec{x} = & \begin{pmatrix} 17 \\ 6 \\ 8 \\ 14 \end{pmatrix} \end{array}$$

$$S = \left\{ \vec{x} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, x_3, x_5 \in \mathbb{R} \right\}$$

Suppose we were handed this solution and asked to check if it solves the system.

Can check that $\begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$ solves $A\vec{x} = \vec{b}$.

Check that $\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ each satisfy $A\vec{x} = \vec{0}$.

This would show each $\vec{x} \in S$ solves $A\vec{x} = \vec{b}$.

$$S' = \left\{ \vec{x} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \quad s,t \in \mathbb{R} \right\}$$

Can also show each $\vec{x} \in S$ solves $A\vec{x} = \vec{b}$.

Are these <u>all</u> of the sol'ns? Row reduction wouldn't lead you to this formula.

Need more theory to prove this gives all solutions.

Fundamental subspaces of a matrix

$A: V \to W$ linear

$\ker A = \operatorname{Null} A := \{ \vec{v} \in V : A\vec{v} = \vec{0} \} \subset V$

$\operatorname{Ran} A := \{ \vec{w} \in W : w = A\vec{v} \text{ for some } \vec{v} \in V \} \subset W$

In other words, $\ker A$ is the set of solutions to the homogeneous eqn $A\vec{v} = \vec{0}$.

$\operatorname{Ran} A$ is the set of $\vec{b}$ for which $A\vec{x} = \vec{b}$ is consistent.

Let A $m \times n$ matrix (i.e. $A: \mathbb{F}^n \to \mathbb{F}^m$).

Any $\vec{w} \in \operatorname{ran} A$ can be written as a linear combination of the cols of A. So $\operatorname{Ran} A$ is sometimes called the <u>column space</u> (when $A: \mathbb{F}^n \to \mathbb{F}^m$)

col A

Can also define $\underbrace{\operatorname{Ran} A^T}_{\text{row space}}$ and $\underbrace{\ker A^T}_{\text{left nullspace}}$.

$$\underbrace{\operatorname{Ran} A,\ \ker A,\ \operatorname{Ran} A^T,\ \ker A^T}_{\text{"the fundamental subspaces"}}$$

<u>Def</u>: Let $A = m \times n$.

$$\text{rank}\, A := \dim(\text{Ran}\, A).$$

<u>Computing fund. spaces and rank</u>

Let A be a matrix, A_e its echelon form.

<u>Thm</u>: 1. The pivot cols of $\underline{\underline{A}}$ gives a basis for $\text{Ran}\, A$.

2. The pivot <u>rows</u> of A_e gives a basis for $\text{Ran}\, A^T$.

3. A basis for $\ker A$ can be found by solving

$A\vec{x} = \vec{0}$.

<u>Ex</u>:

$$\underset{A}{\begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{pmatrix}} \longrightarrow \underset{A_e}{\begin{pmatrix} \boxed{1} & 1 & 2 & 2 & 1 \\ 0 & 0 & \boxed{-3} & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

The pivot columns $\underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \end{pmatrix}}_{\text{from } A}$ form a basis for col A (ran A)

The pivot rows

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ -3 \\ -1 \end{pmatrix}$$

form a basis for the row space $\text{Ran}\, A^T$.

To compute $\ker A$, compute A_{re}

$$A_{re} = \begin{array}{c} \begin{pmatrix} \boxed{1} & 1 & 0 & 0 & 1/3 \\ 0 & 0 & \boxed{1} & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccc} & x_2 & & x_4 & x_5 \end{array} \end{array}$$

x_2, x_4, x_5 free

$$x_1 = -x_2 - \frac{1}{3}x_5$$

$$x_3 = -x_4 - \frac{1}{3}x_5$$

$$\vec{x} = \begin{pmatrix} -x_2 - \frac{1}{3}x_5 \\ x_2 \\ -x_4 - 1/3\, x_5 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1/3 \\ 0 \\ -1/3 \\ 0 \\ 1 \end{pmatrix}$$

form a basis for $\ker A$

no shortcut for finding $\ker A^T$; need to solve $A^T\vec{x} = \vec{0}$

- why does this always give a basis for $\ker A$?

After solving $A\vec{x} = 0$

$$\vec{x} = \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}$$ some entries free, others written in terms of free vars

To make free entries 0, need to set corresponding free var to 0. i.e. the vectors obtained are lin indp. this list of vectors also spans $\ker A$ (it's a complete description for solutions of $A\vec{x} = \vec{0}$). So we have a basis for $\ker A$.

- Pivot cols give basis for ran A:

Notice: pivot cols of A_{re} give a basis for Ran A_e. Why?

Row ops are invertible

$$EA = E\begin{pmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{pmatrix} = \begin{pmatrix} E\vec{a}_1 & \cdots & E\vec{a}_n \end{pmatrix}$$

E inv, so E preserves linear independence

$\rightarrow$ pivot cols of A are lin indp.

$A_{re} = EA$, E invertible.

Let $\vec{v}_1, \ldots, \vec{v}_r$ be the pivot cols of A.

Let $\vec{v}$ be any other col of A.

There are scalars α_i s.t.

$E\vec{v} = \alpha_1 E\vec{v}_1 + \dots + \alpha_r E\vec{v}_r$. why?

$\longrightarrow \vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r$.

So pivot cols span Ran A.

- Ran A^T (row space)
  - the pivot rows of A_e are lin indep

$$\begin{array}{c} w_1 \\ w_2 \\ w_3 \\ w_4 \end{array} \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \end{pmatrix}$$

Let $w_1, \dots, w_r$ be the pivot rows of A_e.

Consider $\alpha_1 w_1 + \dots + \alpha_r w_r = 0$. $\alpha_i \in \mathbb{R}$

α_1 must be 0. α_2 must be 0, and so on.

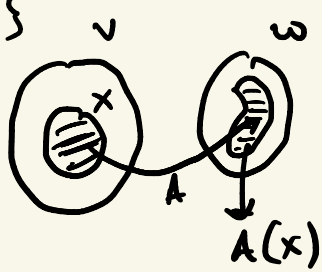
- pivot rows span Ran A^T

claim: row ops do not change row space

For a map A and a set $X \subseteq$ domain A define

$$A(X) := \{ A(x) : x \in X \}$$

"the image of X under A"



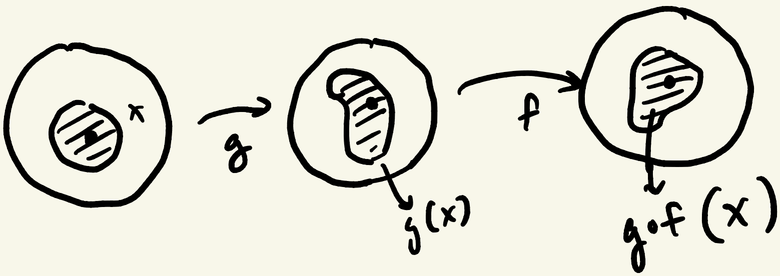
$$A_e = EA, \quad E \ m\times m, \text{ invertible}$$

$$\operatorname{Ran} A_e^T = \operatorname{Ran}((EA)^T)$$

$$= \operatorname{Ran}(A^T E^T) = A^T(\operatorname{Ran} E^T)$$

why?

$$= A^T(\mathbb{R}^m) = \operatorname{Ran} A^T. \quad \square$$



$$\{ f(g(x)) : x \in X \}$$

$$\|$$

$$\{ f(y) : y \in g(X) \}$$

Thm 7.1 (Rank Thm)

$$\text{rank } A = \text{rank } A^T$$

"col rank = row rank"

Pf: rank A = # pivots in A
rank A^T = # pivots in A. $\square$

Thm 7.2 (Rank-nullity thm)

Let $A \in M_{m \times n}$ (i.e. $A : \mathbb{F}^n \to \mathbb{F}^m$, linear). Then

1) $\dim \ker A + \underbrace{\dim \operatorname{Ran} A}_{\text{(rank } A)} = n$ (dimension of domain)

2) $\dim \ker A^T + \text{rank } A = m$

Pf: $\dim \ker A$ = # of free vars
rank A = # of pivots
(# free vars) + (# pivots) = # cols = n.

part (2) is part (1) applied to A^T plus the previous thm.

Ex:

$$\underbrace{\begin{pmatrix} 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 2 & -5 \\ 2 & 2 & 2 & 3 & -8 \end{pmatrix}}_{A} \vec{x} = \underbrace{\begin{pmatrix} 17 \\ 6 \\ 8 \\ 14 \end{pmatrix}}_{\vec{b}}$$

We saw that

$$\vec{x} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

satisfies $A\vec{x} = \vec{b}$. Are these all sol'ns?

these two vectors lie in $\ker A$, are lin indp (why?)

Perform one iteration of "main step" in row red:

$$\text{obtain} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & -3 \\ 0 & 1 & -1 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

already 3 pivots, so $\operatorname{rank} A \geq 3$ (can see it's equal 3, but don't need to know that).

$$\operatorname{rank} A + \dim \ker A = 5 \qquad -\operatorname{rank} A \leq -3$$

$$\Rightarrow \dim \ker A = 5 - \operatorname{rank} A$$

$$\leq 2.$$

So $\ker A$ contains no more than 2 lin indp. we found 2 lin indp vectors in $\ker A$, so we have a basis on our hands.

<u>Thm 7.3</u> $A \in M_{m \times n}$.

Then $A\vec{x} = \vec{b}$ has a solution for each $\vec{b} \in \mathbb{R}^m$ (1)
iff $A^T\vec{x} = 0$ has only the trivial solution. (2)

<u>Pf.</u> Statement (1) $\iff$ A has pivot in each row (i.e. $\operatorname{rank} A = m$). A^T is $n \times m$. Statement (2) $\iff$ A has a pivot in each col (i.e. m pivots).

So (1) and (2) are both equivalent to saying A has m pivots. $\square$

7.4 Completion of an LI list to a basis

The proof for being able to complete a LI list to a basis doesn't provide a practical algorithm to construct the basis.

Notice: if $A_{re} \in M_{m \times n}$ is in reduced ech. form, then the nonzero rows can be completed to a basis of $\mathbb{R}^n$.

$$\text{e.g. } A_{re} = \begin{array}{c} \begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \cdots & \downarrow \end{array} \\ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & & & & 1 \end{pmatrix} \end{array}$$

→ insert three new rows here

$$\begin{array}{c} \begin{array}{ccccccc} * & * & 0 & 0 & * & 0 & * \end{array} \\ \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & & \\ & & & & & & 1 \\ & & & & & & \end{pmatrix} \end{array}$$

if k-th col is not a pivot col, then add $\vec{e}_k^{\,T}$ after row # $k-1$. The new matrix is in echelon form.

i.e. can complete rows to a basis of $\mathbb{R}^n$ by adding standard basis vectors in right spots.

Claim: the same vectors that complete rows of A_{re} to a basis also complete the rows of A to a basis.

Suppose $v_1, v_2, \ldots, v_r$ LI in $\mathbb{R}^n$.

$$A = \begin{pmatrix} v_1^T \\ \vdots \\ v_r^T \end{pmatrix}$$

i-th row of A is v_i

Complete rows of A_{re} to basis of $\mathbb{R}^n$ by adding $v_{r+1}, \ldots, v_n$.

Let $$\tilde{A} := \begin{pmatrix} v_1^T \\ \vdots \\ v_r^T \\ v_{r+1}^T \\ \vdots \\ v_n^T \end{pmatrix}$$

Let $\widetilde{A}_{re}$ be A_{re} with $v_{r+1}^T, \ldots, v_n^T$ added as rows.

$$\widetilde{A}_{re} = E\hat{A}, \quad E = \text{product of elementary matrices}$$

$$\hat{A} = \underset{\text{inv}}{\underset{\downarrow}{E^{-1}}}\ \underset{\text{inv}}{\underset{\downarrow}{\widetilde{A}_e}}, \quad \text{so } \hat{A} \text{ invertible.}$$

$\Downarrow$

rows of $\hat{A}$ form basis of $\mathbb{R}^n$.

8. Matrix of a linear map enhanced

Change of coords

- For $A: \mathbb{R}^n \to \mathbb{R}^m$ linear, the matrix $[A]$ and A itself were viewed as the same
- For general $A: V \to W$, must be careful

Let V be a vector space, $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ basis for V.

For each $\vec{v} \in V$, there is a <u>unique</u> n-tuple of numbers $(x_1, \dots, x_n)$ s.t.

$$v = x_1 \vec{b}_1 + \dots + x_n \vec{b}_n = \sum_{k=1}^{n} x_k \vec{b}_k$$

$(x_1, \dots, x_n)$ are called the <u>coordinates of v</u> wrt $\mathcal{B}$.

$$[\vec{v}]_{\mathcal{B}} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \ (\text{or } \mathbb{C}^n)$$

<u>coordinate vector</u>

Note: the map $\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}}$ is an isomorphism

In fact, it is the unique isomorphism s.t.

$$C : V \to \mathbb{R}^n$$

$$C(\vec{b}_i) = \vec{e}_i, \quad i = 1, \dots, n.$$

<u>Def:</u> Let $T : V \to W$ linear.

$\mathcal{A} = \{\vec{a}_1, \dots, \vec{a}_n\}$ basis for V

$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_m\}$ basis for W

The <u>matrix</u> of T wrt $\mathcal{A}, \mathcal{B}$ is the matrix $[T]_{\mathcal{B}\mathcal{A}}$ whose k-th col is

$$[T\vec{a}_k]_{\mathcal{B}}$$

Let $\vec{v} \in V$.

$$\vec{v} = [\vec{v}]_{\mathcal{A},1}\vec{a}_1 + \dots + [\vec{v}]_{\mathcal{A},n}\vec{a}_n$$

$$= \sum_{k=1}^{n} [\vec{v}]_{\mathcal{A},k}\vec{a}_k$$

$$T\vec{v} = \sum_{k=1}^{n} [\vec{v}]_{\mathcal{A},k} T\vec{a}_k$$

$$[T\vec{v}]_{\mathcal{B}} = \left[\sum_{k=1}^{n} [\vec{v}]_{\mathcal{A},k} T\vec{a}_k\right]_{\mathcal{B}}$$

$$= \sum_{k=1}^{n} [\vec{v}]_{\mathcal{A},k} \, [T\vec{a}_k]_{\mathcal{B}}$$

$$= [T]_{\mathcal{B}\mathcal{A}} [\vec{v}]_{\mathcal{A}}.$$

$$\boxed{[T\vec{v}]_{\mathcal{B}} = [T]_{\mathcal{B}\mathcal{A}} [\vec{v}]_{\mathcal{A}}}$$

Suppose $T_1 : X \to Y$, $T_2 : Y \to Z$ linear maps

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ bases for X, Y, Z resp. $\mathcal{A} = \{a_1, \dots, a_\ell\}$

$$[T_2 \underbrace{T_1 a_i}_{\in Y}]_{\mathcal{C}} = [T_2]_{\mathcal{C}\mathcal{B}} [T_1 a_i]_{\mathcal{B}}$$

so i-th col of $T_2 T_1$ is exactly

$[T_2]_{\mathcal{C}\mathcal{B}}$ times $[T_1 a_i]_{\mathcal{B}\mathcal{A}}$

$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad$ i-th col of $[T_1]_{\mathcal{B}A}$

$$\Rightarrow \boxed{[T_2 T_1]_{\mathcal{C}\mathcal{A}} = [T_2]_{\mathcal{C}\mathcal{B}} [T_1]_{\mathcal{B}\mathcal{A}}} \quad (*)$$

8.3 Change of coords

$\mathcal{A} = \{\vec{a}_1, \ldots, \vec{a}_n\}$, $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ bases of V.

Let $I: V \to V$ be the identity.

$[I]_{\mathcal{B}\mathcal{A}}$ is not necessarily the identity matrix.

if $\mathcal{B} = \mathcal{A}$, then it is, but otherwise no.

$[I]_{\mathcal{B}\mathcal{A}}$ takes a vector in $\mathcal{A}$ coords and produces coords in $\mathcal{B}$.

$[I]_{\mathcal{B}\mathcal{A}}$ has $[\vec{a}_k]_{\mathcal{B}}$ as its k-th col.

Obs: $[I]_{\mathcal{B}\mathcal{A}} = ([I]_{\mathcal{A}\mathcal{B}})^{-1}$ (apply $*$)

Ex (standard basis:

Let $V = \mathbb{R}^n$. $S = \{\vec{e}_1, \ldots, \vec{e}_n\}$

$\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$

$[I]_{S\mathcal{B}} \longrightarrow$ k-th col is $[I\vec{b}_k]_S = [\vec{b}_k]_S$

$= \vec{b}_k$

$[I]_{S\mathcal{B}}$ is the matrix $B = [\vec{b}_1, \ldots, \vec{b}_n]$

$$[I]_{\mathcal{B}S} = B^{-1}$$

e.g. $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

$$[I]_{S\mathcal{B}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ has coordinates $(1, 0)$ wrt $\mathcal{B}$

$$[I]_{S\mathcal{B}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

In $\mathbb{R}^n$, the coordinates of any vector wrt standard basis are just the entries of the vector itself

$$[I]_{\mathcal{B}S} = [I]_{S\mathcal{B}}^{-1} = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$\frac{1}{3} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3} \left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$$

i.e. $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

Ex: $\mathbb{P}_1$

$$\mathcal{A} = \{1, 1+x\}, \quad \mathcal{B} = \{1+2x, 1-2x\}$$

$$S = \{1, x\}$$

to compute $[I]_{\mathcal{B}\mathcal{A}}$, use $[I]_{\mathcal{B}\mathcal{A}} = [I]_{\mathcal{B}S}[I]_{S\mathcal{A}}$

$$[I]_{S\mathcal{A}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: A \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

$$[I]_{S\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} =: B \qquad = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$$

$$[I]_{\mathcal{B}\mathcal{A}} = [I]_{\mathcal{B}S}[I]_{S\mathcal{A}} = B^{-1}A = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$[I]_{\mathcal{A}\mathcal{B}} = A^{-1}B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

Ex: $V = \mathbb{P}_3$ $\quad S = \{1, x, x^2, x^3\}$

$T : \mathbb{P}_3 \longrightarrow \mathbb{P}_3$

$T(p) = p'$

$$[T]_{SS} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array}$$

$T(1) = 0$

$T(x) = 1$

$T(x^2) = 2x$

$T(x^3) = 3x^2$

Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$[p]_S = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$[T(p)]_S = [T]_{SS}[p]_S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{pmatrix} \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array} \longrightarrow a_1 + 2a_2 x + 3a_3 x^2 + 0x^3$$

Ker T? Ran T?

Matrix of a transformation and change of coords

$T: V \to W$ linear

$\mathcal{C}, \tilde{\mathcal{C}}$ bases of V

$\mathcal{B}, \tilde{\mathcal{B}}$ bases of W

$$\boxed{[T]_{\tilde{\mathcal{B}}\tilde{\mathcal{C}}} = [I]_{\tilde{\mathcal{B}}\mathcal{B}} [T]_{\mathcal{B}\mathcal{C}} [I]_{\mathcal{C}\tilde{\mathcal{C}}}}$$

$$\underset{\mathcal{C}}{T: V} \longrightarrow \underset{\mathcal{B}}{W}$$

$S := [\cdot]_{\mathcal{C}} : V \to \mathbb{R}^n$ iso

$R := [\cdot]_{\mathcal{B}} : W \to \mathbb{R}^m$ iso

$R \circ T \circ S^{-1} : \mathbb{R}^n \to \mathbb{R}^m$

i.e. $\underset{A}{\underset{\shortparallel}{RTS^{-1}}}$ is an $m \times n$ matrix

Rank thm $\Longrightarrow$ $\dim \ker A + \dim \operatorname{Ran} A = n$

Note: $T: V \to W$, $S: U \xrightarrow{\cong} V$

Then $\ker T \cong \ker TS$

$\operatorname{ran} T \cong \operatorname{ran} TS$

Let $v_1, \ldots, v_r$ be a basis for $\ker T$.

$S^{-1}v_1, \ldots, S^{-1}v_r$ each lie in $\ker TS$

They are lin indp.

Let $u \in \ker TS$. Then $TSu = \vec{0}$

$T(Su) = \vec{0}$

$Su \in \ker T$. $Su = \alpha_1 v_1 + \cdots + \alpha_r v_r$

$u = \alpha_1 S^{-1}v_1 + \ldots + \alpha_r S^{-1} v_r$

$u \in \operatorname{span} \{ S^{-1} v_i \}_{i=1,\ldots,r}$

Do similar thing for range.

Conclude $\dim \ker A = \dim \ker T$

$\dim \operatorname{ran} T = \dim \operatorname{Ran} A$.

So rank-nullity thm works for general

$T: V \to W$, W, V f.d vs

$\mathcal{C} = \{\vec{a}_1, \dots, \vec{a}_n\}$ $\quad T: V \to V$ linear

$[T]_{\mathcal{C}\mathcal{C}}$ same basis for "inputs" and "outputs"

$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ another basis

$$[T]_{\mathcal{B}\mathcal{B}} = [I]_{\mathcal{B}\mathcal{C}} [T]_{\mathcal{C}\mathcal{C}} [I]_{\mathcal{C}\mathcal{B}}$$

$$Q := [I]_{\mathcal{C}\mathcal{B}}$$

$$[T]_{\mathcal{B}\mathcal{B}} = Q^{-1} [T]_{\mathcal{C}\mathcal{C}} Q$$

Def: A matrix A is <u>similar</u> to a matrix B if there is an invertible Q s.t.

$$A = Q^{-1} B Q.$$

Note: similarity is an equivalence relation.

1) A is similar to itself
2) A sim $B \implies B$ sim A
3) A sim B, B sim $C \implies A$ sim C

Similar matrices can be thought of as different matrix representations of the same operator T.