A transformation T from a set X to a set Y is a rule that assigns to each $x \in X$ an output $y = T(x) \in Y$. (we write $T: X \to Y$)

X is called the domain. Y is the target space or codomain.

Def: Let V, W be vector spaces over a field $\mathbb{F}$. $T: V \to W$ is linear if

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall u, v \in V$

2) $T(\alpha\vec{v}) = \alpha T(\vec{v}) \quad \forall v \in V, \forall \alpha \in \mathbb{F}$.

equivalently,

$$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v}) \quad \forall u, v \in V \quad \forall \alpha, \beta \in \mathbb{F}$$

Ex: $V = \mathbb{P}_n$ (polynomials of degree $\le n$)

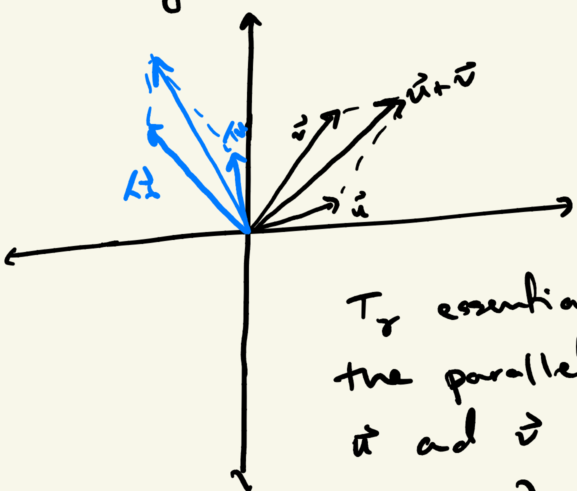
$W = \mathbb{P}_{n-1}$

Define $T(p) = p'$.

Ex: $V = W = \mathbb{R}^2$.

Let $T_\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by γ radians counterclockwise.

Why is T_γ linear?



$\gamma = \pi/2$

T_γ essentially "picks up" the parallelogram formed by $\vec{u}$ and $\vec{v}$ and rotates it, so $T(\vec{u}+\vec{v})$ and $T\vec{u} + T\vec{v}$ end up being the same.

Also not hard to see $\alpha T\vec{u} = T(\alpha\vec{u})$

Ex.: $V = W = \mathbb{R}^2$.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be reflection about x-axis.

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}.$$

Can check algebraically that T is linear.

Ex.: Recall that $\mathbb{R}$ is a vector space over $\mathbb{R}$ itself. (A bit weird since vectors and scalars are one and the same)

Suppose $T: \mathbb{R} \to \mathbb{R}$ is linear.

Let $x \in \mathbb{R}$. Then $T(x) = T(x \cdot 1) = x\,T(1)$.

Let $a = T(1)$. Then $T(x) = ax$.

So T is the map that scales by a.

3.2 Linear transformations $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

$\mathbb{R}^n$ has standard basis vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

n-th spot

Let $\vec{a}_i = T(e_i)$, $i = 1, \ldots, n$.

__Claim:__ The values of $\vec{a}_i$ determine T completely.

__Pf:__ Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Then $\vec{x} = x_1 \vec{e}_1 + \cdots + x_n \vec{e}_n$.

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) \\ &= x_1 T(\vec{e}_1) + \cdots + x_n T(\vec{e}_n) \\ &= x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n \end{aligned}$$

Define $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$ ($\vec{a}_k$ forms the k-th column)

$\boxed{\text{"A contains all information about T"}}$

Write A as

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

With this notation,

$$\vec{a}_k = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ a_{m,k} \end{pmatrix}$$

How should we define multiplication of a matrix and a vector?

We want $A\vec{x} = T(\vec{x})$

$$T(\vec{x}) = \sum_{k=1}^{n} x_k \vec{a}_k = x_1 \begin{pmatrix} a_{1,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ \vdots \\ a_{m,2} \end{pmatrix} \qquad (*)$$
$$+ \cdots + x_n \begin{pmatrix} a_{1,n} \\ \vdots \\ a_{m,n} \end{pmatrix}$$

This leads to <u>define</u> $A\vec{x}$ as the thing on the right hand side.

$$\boxed{\text{multiply each column of } A \text{ by the corresponding entry of } \vec{x} \text{ and add them up}}$$

<u>Ex:</u>

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 14 \\ 10 \end{pmatrix}.$$

Notice in (*) that if we collect the terms then the k-th entry of Ax is

$$a_{k,1}x_1 + a_{k,2}x_2 + \cdots + a_{k,n}x_n$$

$$= \sum_{j=1}^{n} a_{k,j}\, x_j \qquad (k = 1, \ldots, m)$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 3 \\ 3\cdot 1 + 2\cdot 2 + 1\cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 14 \\ 10 \end{pmatrix}$$

3.3 Linear maps and generating sets

Prop: Let $T: V \to W$ be linear. Then T is completely defined by its values on a generating set.

Put differently, if $T_1, T_2 : V \to W$ agree on a generating set, then $T_1 = T_2$ (they agree everywhere).

Pf: ... $\square$

Conclusions:

- To get the matrix of $T: \mathbb{R}^n \to \mathbb{R}^m$, form the matrix with columns $\vec{a}_k = T\vec{e}_k$.

- If we have the matrix A of T, then $T(\vec{x})$ can be computed as

$$T(\vec{x}) = A\vec{x}$$

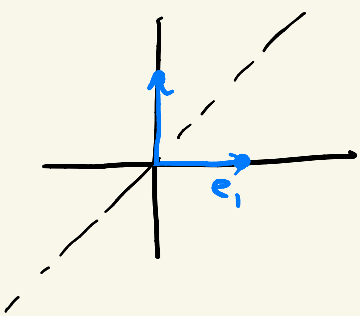
↳ can compute "coord. by coord" or "row by column"

The matrix of $T: \mathbb{R}^n \to \mathbb{R}^m$ is sometimes denoted $[T]$. Sometimes, we drop the brackets and conflate T w/ its matrix.

Sometimes write $T\vec{u}$ instead of $T(\vec{u})$.

Rmk: In an expression like $A\vec{x}$, # of cols of A must match # of entries in $\vec{x}$.

Q: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be reflection about the line $x_1 = x_2$ (i.e. the line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$). Find the matrix of T.



$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

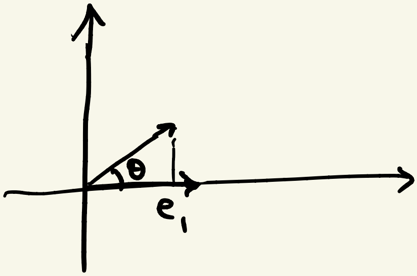
$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

So $[T] = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

So $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}0\\1\end{pmatrix} + y\begin{pmatrix}1\\0\end{pmatrix}$

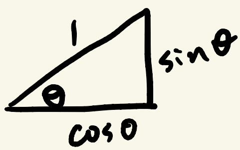
$= \begin{pmatrix}y\\x\end{pmatrix}$ ✓

Q: Let $\theta \in \mathbb{R}$ and T_θ be rotation CCW by θ. What is $[T_\theta]$?

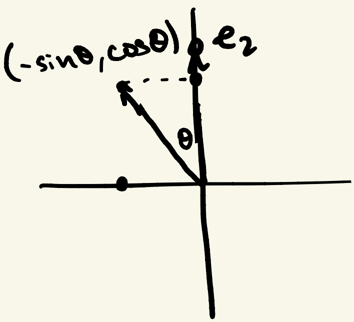


$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$



$$[T_\theta] = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$



$$[T_{\pi/4}] = \begin{pmatrix}\cos\pi/4 & -\sin\pi/4\\ \sin\pi/4 & \cos\pi/4\end{pmatrix}$$

$$= \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$$

Linear Transformations as a vector space

$T: V \to W$ linear, α scalar

Can define αT to be the map given by

$$(\alpha T)(\vec{v}) = \alpha (T\vec{v}) \quad \forall \vec{v} \in V$$

can check that αT defined this way is also linear:

$$\begin{aligned}
&(\alpha T)(\beta_1 \vec{v}_1 + \beta_2 \vec{v}_2) \\
&\overset{\text{def}}{=} \alpha\left(T(\beta_1 \vec{v}_1 + \beta_2 \vec{v}_2)\right) \\
&\overset{T \text{ is linear}}{=} \alpha(\beta_1 T\vec{v}_1 + \beta_2 T\vec{v}_2) \\
&= \alpha\beta_1 T\vec{v}_1 + \alpha\beta_2 T\vec{v}_2 \\
&= \beta_1(\alpha T\vec{v}_1) + \beta_2(\alpha T\vec{v}_2)
\end{aligned}$$

Given $T_1, T_2 : V \to W$ both linear, can define $T_1 + T_2 : V \to W$ by

$$(T_1 + T_2)(\vec{v}) = T_1 \vec{v} + T_2 \vec{v}$$

$T_1 + T_2$ is also linear for similar reasons.

With these operations, the space $\mathcal{L}(V, W)$ of all linear maps from V to W is itself a vector space!

<u>Pf:</u> tedious. $\square$

Composition of linear transformations and matrix multiplication

---

we have defined multiplication of a matrix and column.

Can extend this to define product of two matrices.

Def: if $b_1, b_2, \dots, b_r$ are the columns of B, then AB is defined to be the matrix

$$AB := \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_r \\ | & | & & | \end{bmatrix}$$

i.e. the k-th col is Ab_k

If we use the row-by-column rule, we find that

$$(AB)_{j,k} = (\text{row } \# j \text{ of } A) \cdot (\text{column } \# k \text{ of } B)$$

$$= \sum_{l} a_{j,l} b_{l,k}$$

Notice that for AB to be well-defined, the size of a row in A must match the size of a col in B.

$$\underset{m \times n}{A} \quad \underset{n \times r}{B} \longrightarrow \underset{m \times r}{AB}$$

dimensions must match

Ex: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 1 & 1 \\ 3 & 3 \end{pmatrix} \rightsquigarrow$ Result is 2×2

2×3 $\quad$ 3×2

$$= \begin{pmatrix} 1\cdot 5 + 2\cdot 1 + 3\cdot 3 & 1\cdot 6 + 2\cdot 1 + 3\cdot 3 \\ 4\cdot 5 + 5\cdot 1 + 6\cdot 3 & 4\cdot 6 + 5\cdot 1 + 6\cdot 3 \end{pmatrix}$$

why is multiplication defined like this?

Suppose $T_1 : \mathbb{R}^n \to \mathbb{R}^m$, $T_2 : \mathbb{R}^r \to \mathbb{R}^n$ linear

- Define $T = T_1 \circ T_2$ by

$$(T_1 \circ T_2)(x) = T_1(T_2(x)) \qquad \mathbb{R}^r \xrightarrow{T_2} \mathbb{R}^n \xrightarrow{T_1} \mathbb{R}^m, \quad x \in \mathbb{R}^r$$

- check that $T_1 \circ T_2$ is a linear map from $\mathbb{R}^r$ to $\mathbb{R}^m$.

- what is the matrix of $T = T_1 \circ T_2$?

Let A be the matrix of T_1, B the matrix of T_2

By definition the columns of the matrix of T are

$$T(\vec{e}_1), \dots, T(\vec{e}_r)$$

$$\rightsquigarrow T(\vec{e}_k) = (T_1 \circ T_2)(\vec{e}_k)$$
$$= T_1(T_2(\vec{e}_k))$$
$$= T_1(B\vec{e}_k)$$
$$= T_1(\vec{b}_k)$$
$$= A\vec{b}_k.$$

multiply B by $\vec{e}_k$ extracts k-th column

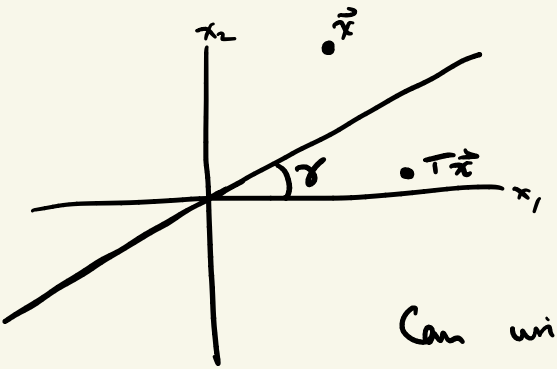
So the columns of T's matrix are $A\vec{b}_1, \dots, A\vec{b}_r$!

In other words, matrix multiplication is defined so that the matrix of $T_1 \circ T_2$ is simply AB.

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$ reflection about $x_1 = 3x_2$

Find $[T]$

↳ the matrix of T



Can write T as composition of simpler maps.

Let T_0 = reflection across x_1-axis.

Let γ = angle of line to $+x_1$-axis

$$T = R_\gamma \circ T_0 \circ R_{-\gamma}$$

(note the order)

By discussion above,

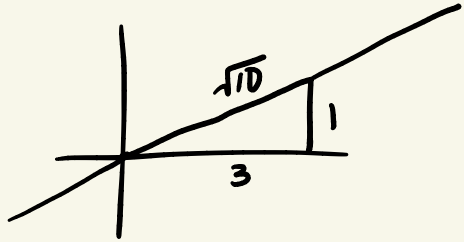
$$[T] = [R_\gamma][T_0][R_{-\gamma}]$$

$$[T_0] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$R_\gamma = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$$

$$R_{-\gamma} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}$$

$\cos\gamma$?
$\sin\gamma$?



$$\cos\gamma = \frac{3}{\sqrt{10}}, \quad \sin\gamma = \frac{1}{\sqrt{10}}$$

$$[T] = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{10}}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

$$= \frac{1}{10}\left(\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

$$= \frac{1}{10}\left(\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}\right)\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 8 & 6 \\ 6 & -8 \end{pmatrix} = \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix}$$

Properties

1) $A(BC) = (AB)C$ associative
provided one side is defined

2) $A(B+C) = AB + AC$

$(A+B)C = AC + BC$
provided one side is defined

3) $\alpha(AB) = (\alpha A)B = A(\alpha B)$

But, $AB \neq BA$ usually!

$A = m \times n$
$B = n \times r$ $\rightsquigarrow$ AB makes sense
BA is not even defined.

Even if A and B are both $n \times n$,
$AB \neq BA$ most of the time.

Fact: $$(AB)^T = B^T A^T.$$

check this (if you feel like it)

Def.: Let $A = n \times n$ (square matrix).

The trace of A is

$$\operatorname{tr} A = \sum_{k=1}^{n} a_{k,k}.$$

i.e. sum of the diagonal elements

Thm: Let $A = m \times n$, $B = n \times m$. Then

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

Pf:

$$T_1 : M_{n \times m} \longrightarrow \mathbb{R}$$

$$T_2 : M_{n \times m} \longrightarrow \mathbb{R}$$

$$T_1(X) = \operatorname{tr}(AX)$$

$$T_2(X) = \operatorname{tr}(XA)$$

Let $X_{i,j}$ be the $n \times m$ matrix $= 0$ everywhere except at i,j entry, where it is 1.

$$T_1(X_{i,j}) = \operatorname{tr}(AX_{i,j}) \qquad \begin{array}{c c} & \begin{array}{ccc} & j & \\ \end{array} \\ i \rightarrow & \begin{pmatrix} - & 0 & - \\ - & 0 & - \\ - & 1 & - \\ - & 0 & - \end{pmatrix} \end{array}$$

$$= a_{j,i}.$$

$$A\vec{e}_i = \vec{a}_i$$

$$A \begin{array}{c} \begin{array}{ccccc} & & j & & \end{array} \\ \begin{pmatrix} | & | & & | & & | \\ \vec{0} & \vec{0} & \cdots & \vec{e}_i & \cdots & \vec{0} \\ | & | & & | & & | \end{pmatrix} \end{array} = \begin{pmatrix} | & | & & | & & | \\ \vec{0} & \vec{0} & \cdots & A\vec{e}_i & \cdots & \vec{0} \\ | & | & & | & & | \end{pmatrix}$$

$$= \begin{array}{c} \begin{array}{ccccc} & & j & & \end{array} \\ \begin{pmatrix} | & | & & | & & | \\ 0 & 0 & \cdots & \vec{a}_i & \cdots & 0 \\ | & | & & | & & | \end{pmatrix} \end{array}$$

$$T_2(X_{i,j}) = \operatorname{tr}(X_{i,j}A)$$

$$\begin{array}{c c} & j \\ i & \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} \end{array} \vec{a}_k = \begin{array}{c c} & \\ i & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_{j,k} \\ \vdots \\ 0 \end{pmatrix} \end{array}$$

$$a_{j,k}$$

$$X_{i,j}A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & \end{array} \\ i & \begin{pmatrix} \ddots & & & \\ a_{j,1} & a_{j,2} & \cdots & a_{j,n} \\ & & & \end{pmatrix} \end{array}$$

$$\operatorname{tr}(X_{i,j}A) = a_{j,i}. \qquad \square$$

Invertible transformations

V = vector space

$I_V : V \to V$ is the map

$$I_V(\vec{x}) = \vec{x} \quad \forall \vec{x} \in V.$$

Note I_V is linear.

I_V is called the <u>identity transformation</u> of V.

often simply write I

- $I : \mathbb{R}^n \to \mathbb{R}^n$. The matrix of I

$$I = I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

for any A, $AI = A$, $IA = A$
(when defined)

<u>Def.</u> Let $A : V \to W$ be linear. A is <u>left-invertible</u> if there is some $B : W \to V$ s.t.

$$BA = I_V$$

A is <u>right-invertible</u> if there is some $C : W \to V$ s.t.

$$AC = I_W$$

- generally, left/right inverses are not unique

Def: $A: V \to W$ is <u>invertible</u> if it is both left and right invertible.

Thm: If $A: V \to W$ is invertible, then its left and right inverses B and C are unique and coincide.

Corollary: $A: V \to W$ (linear) is invertible iff there is a unique linear transf $A^{-1}: W \to V$ s.t.

$$A^{-1}A = I_V, \quad AA^{-1} = I_W.$$

A^{-1} is called the <u>inverse</u> of A.

Pf of thm: Suppose $BA = I$, $AC = I$.

$$BAC = B(AC) = BI = B.$$

o.t.o.h.,

$$BAC = (BA)C = IC = C.$$

So $B = C$.

If $B_1 A = I$ also, then above tells us $B_1 = C$, so $B = B_1$. Similarly for C. $\square$

Def: A matrix is invertible if the corresponding linear transformation is.

Ex: 1) I is invertible. $I^{-1} = I$.

2) rotation R_γ is invertible

$$(R_\gamma)^{-1} = R_{-\gamma}.$$

can also check this by matrix mult.

3) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is left invertible

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} = I_1$$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not right invertible since it has multiple left inverses

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - 1 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$$

why does this imply $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not right inv?

Rmk: we will show later that an invertible matrix must be square $(n \times n)$ and that if A is square and has either a left inv or right inv, then A is invertible.

Thm: If A and B are invertible, and AB is defined, then AB is invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Pf: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1}$

$= AA^{-1} = I.$

Also $(B^{-1}A^{-1})(AB) = B^{-1}IB = B^{-1}B = I.$ $\square$

Rmk: It's possible that AB is invertible, but A and B are not individually.

e.g.
$$\underset{A}{\overset{2\times 3}{\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}}} \underset{B}{\overset{3\times 2}{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}} = \underset{AB}{\overset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}$$

Thm 6.5 (Inverse of A^T)

If A is invertible, then so is A^T, and

$$(A^T)^{-1} = (A^{-1})^T.$$

Pf: $$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I. \quad \square$$

Def: An isomorphism is an invertible linear transformation.

If $A: V \to W$ is an isomorphism, V and W are called isomorphic (write $V \cong W$)

If V and W are isomorphic, they are "the same" in some sense.

Thm: Let $A : V \longrightarrow W$ be an isomorphism. Let $\vec{v}_1, \ldots, \vec{v}_n$ be a basis in V. Then $A\vec{v}_1, \ldots, A\vec{v}_n$ is a basis in W.

Pf:

linear independence:

$$\beta_1(A\vec{v}_1) + \ldots + \beta_n(A\vec{v}_n) = \vec{0} \in W$$

Linearity $\Longrightarrow$ $A(\beta_1\vec{v}_1 + \ldots + \beta_n\vec{v}_n) = \vec{0}$

A has an inverse A^{-1}

$$A^{-1}\left(A(\beta_1\vec{v}_1 + \ldots + \beta_n\vec{v}_n)\right) = A^{-1}\vec{0}$$

$$= I_V(\beta_1\vec{v}_1 + \ldots + \beta_n\vec{v}_n) = 0$$

$$\Longrightarrow \beta_1\vec{v}_1 + \ldots + \beta_n\vec{v}_n = 0$$

$$\Longrightarrow \beta_i = 0 \ \forall i.$$

spanning: Let $\vec{w} \in W$. Then $A^{-1}\vec{w} \in V$.

There are $\alpha_i \in \mathbb{F}$ s.t.

$$\alpha_1\vec{v}_1 + \ldots + \alpha_n\vec{v}_n = A^{-1}\vec{w}.$$

$$\Longrightarrow \quad A(\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n) = \vec{w}$$

$$\Longrightarrow \quad \alpha_1 A\vec{v}_1 + \dots + \alpha_n A\vec{v}_n = \vec{w}. \quad \square$$

<u>Thm:</u> Let $A: V \to W$ be linear. If $\vec{v}_1, \dots, \vec{v}_n$ is a basis for V, and $\vec{w}_1, \dots, \vec{w}_n$ is a basis for W, and $A\vec{v}_k = \vec{w}_k$ for $k = 1, \dots, n$, then A is an isomorphism.

<u>Pf:</u> Define the linear map

$$B: W \to V \quad \text{by} \quad B(\vec{w}_k) = \vec{v}_k, \; k = 1, \dots, n.$$

This map is completely determined by this assignment since $\vec{w}_1, \dots, \vec{w}_n$ is a basis.

Check that $B = A^{-1}$. $\square$

e.g. 1) $A: \mathbb{R}^{n+1} \longrightarrow \mathbb{P}_n^{\mathbb{R}}$

$$A(\vec{e}_1) = 1, \quad A(\vec{e}_2) = t, \ \ldots, \ A(\vec{e}_{n+1}) = t^n$$

A is an iso. by thm above

- set $n=2$

$$A\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = 3 + t + 5t^2$$

2) Let V be a v.s. over $\mathbb{R}$ with basis $\vec{v}_1, \ldots, \vec{v}_n$. Define

$$A: \mathbb{R}^n \longrightarrow V$$
$$A\vec{e}_k = \vec{v}_k.$$

A is an iso.

So $V \cong \mathbb{R}^n$.