Final Exam/Homework

Exercise 1

Let $T : \mathbb{P}_n \to \mathbb{P}_n$ be the linear transformation defined by $T(p(t)) = 2p(t) + 3p'(t) - 4p''(t)$. Find the matrix of T with respect to the basis $1, t, t^2, \ldots, t^n$ of $\mathbb{P}_n$. You can express the matrix by specifying what each $a_{i,j}$ is, where $a_{i,j}$ is the entry in the i-th row and j-th column of the matrix (or whatever is easiest to write, as long as you give a complete description).

Exercise 2

Find the change of coordinates matrix from the basis 1, $1+t$ to the basis $1-t$, $2t$ in $\mathbb{P}_1$.

Exercise 3

Let $Perm(n)$ be the set of permutations of $\{1, 2, \ldots, n\}$. Let $\tau \in Perm(n)$. Show that the map from $Perm(n)$ to $Perm(n)$ defined by $\sigma \mapsto \sigma\tau$ is a bijection (one-to-one and onto). In other words, composition with a fixed permutation τ takes the list of all permutations and rearranges it, but does not duplicate or remove any (it permutes the permutations). **Hint:** Use the fact that $Perm(n)$ is a group, where the group operation is composition of permutations. In particular, you can take inverses (i.e. τ has some invedrse τ^{-1} such that $\tau\tau^{-1} = \text{id}$, where id is the identity permutation).

Exercise 4

Show that the determinant of a matrix is the product of its eigenvalues (counting multiplicities). **Hint:** First show that the characteristic polynomial $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$ (note the λ_i's occur before λ), where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (with multiplicity). Then plug in $\lambda = 0$. Note that in general, a polynomial of degree n can be written as $p(\lambda) = c(\lambda - r_1)(\lambda - r_2) \cdots (\lambda - r_n)$ (note the λ comes before each λ_i), so the first step is to show that this c is $(-1)^n$. If you use the permutation definition of the determinant, there is exactly one term that contributes the λ^n part of the characteristic polynomial, and it comes from the identity permutation; expand this term to see that the coefficient of λ^n is $(-1)^n$.

Exercise 5

Suppose the entries of A and A^{-1} are integers. What are the possible values of $\det(A)$? **Hint:** Use the fact that $\det(A) \cdot \det\left(A^{-1}\right) = 1$.

Exercise 6

A matrix A is called *nilpotent* if $A^k = 0$ for some positive integer k. Show that the determinant of a nilpotent matrix is 0.

Exercise 7

How are the determinants of A and B related if

(a)

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, B = \begin{pmatrix} 2a_1 & 3a_2 & 5a_3 \\ 2b_1 & 3b_2 & 5b_3 \\ 2c_1 & 3c_2 & 5c_3 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, B = \begin{pmatrix} 3a_1 & 4a_2 + 5a_1 & 5a_3 \\ 3b_1 & 4b_2 + 5b_1 & 5b_3 \\ 3c_1 & 4c_2 + 5c_1 & 5c_3 \end{pmatrix}$$

Exercise 8

Recall that the Fibonacci sequence is defined by $\varphi_0 = 0$, $\varphi_1 = 1$, and $\varphi_{n+2} = \varphi_{n+1} + \varphi_n$ for $n \geq 0$ (the first few terms are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$). You will now find a formula for the n-th Fibonacci number.

(a) Find a 2×2 matrix A such that

$$\begin{pmatrix} \varphi_{n+2} \\ \varphi_{n+1} \end{pmatrix} = A \begin{pmatrix} \varphi_{n+1} \\ \varphi_n \end{pmatrix}.$$

(b) Diagonalize A and find a formula for A^n.

(c) Noticing that

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi_n \end{pmatrix} = A^n \begin{pmatrix} \varphi_1 \\ \varphi_0 \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

use your formula for A^n to find a formula for φ_n.

Exercise 9

Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$. Show that $D(\mathbf{v}_1, \mathbf{v}_2) > 0$ if and only if there is some rotation R such that $R\mathbf{v}_1$ is a positive multiple of $\mathbf{e}_1$ and $R\mathbf{v}_2$ lies in the upper half plane (i.e. has positive y-coordinate). This illustrates that the sign of $\det A$ says something about preservation/reversal of orientation. **Hint:** Note that the determinant of a rotation is 1. Also, what is the determinant of

$$\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}?$$

(Or more generally, what if you replace the 1 with any positive number?)