HW1

Exercise 1. Let $\mathbf{x} = (1,2,3)^T$, $\mathbf{y} = (y_1, y_2, y_3)^T$, and $\mathbf{z} = (4,2,1)^T$. Compute $2\mathbf{x}$, $3\mathbf{y}$, and $\mathbf{x} + 2\mathbf{y} - 3\mathbf{z}$.

Exercise 2. True or false (and give a brief justification):

(a) Every vector space contains a zero vector.

(b) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.

(c) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree less than or equal to n.

Exercise 3. True or false (and give a brief justification):

(a) Any set of vectors that contains the zero vector is linearly dependent.

(b) A basis must contain the zero vector.

(c) Subsets of a linearly dependent set are linearly dependent.

(d) Subsets of a linearly independent set are linearly independent.

Exercise 4. A matrix A is said to be **symmetric** if $A^T = A$. Write down a basis for the space of 2×2 symmetric matrices (there are many possible answers). How many elements does this basis have?

Exercise 5. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in a vector space V. Is it possible that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, but $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$, and $\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$ are linearly *independent*? Justify your answer.

Exercise 6. Multiply the following:

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix}$

Exercise 7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by reflection about the line $x_2 = x_1$ (or $y = x$ if one prefers). Note that this map is linear. Find the matrix of T.

Exercise 8. Find the 3×3 matrix that represent the following linear transformations of $\mathbb{R}^3$:

(a) Projection onto the x-y plane.

(b) Reflection about the x-y plane.

(c) Rotation 30° counterclockwise about the z-axis.

Exercise 9. Multiply two rotation matrices T_α and T_β (this is a rare instance where the order of multiplication does not matter since rotations commute). Use this to deduce formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ in terms of $\cos(\alpha)$, $\sin(\alpha)$, $\cos(\beta)$, and $\sin(\beta)$.

Exercise 10. Find all left inverses of the column vector $(1, 2, 3)^T$.

Exercise 11. Is the column $(1, 2, 3)^T$ right invertible? Justify your answer.

Exercise 12. Suppose $AB = \mathbf{0}$ for some matrices A and B. Is it possible that A is invertible? Justify your answer.

Exercise 13. Let X be a subspace of a vector space V. Let $\mathbf{v} \in V \setminus X$ (that is, $\mathbf{v}$ lies V but not in X). Let $\mathbf{x} \in X$. Show that $\mathbf{x} + \mathbf{v}$ is *not* in X.