HW3 All of the following exercises are from LADW.

Exercise 5.4. (An old problem revisited: now this problem should be easier) Is it possible that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, but the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$ are linearly *independent*? **Hint:** What dimension can the subspace $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ have?

Exercise 5.5. Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be a basis in V. Show that $\mathbf{u} + \mathbf{v} + \mathbf{w}$, $\mathbf{v} + \mathbf{w}$, $\mathbf{w}$ is also a basis in V. **Hint:** You can form a matrix whose columns are coordinate vectors with respect to the basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

Exercise 6.1. True or false (explain your answers):

(a) Any homogeneous system of linear equations has at least one solution;

(b) Any system of n linear equations in n unknowns has at least one solution;

(c) The solution set of any system of m equations in n unknowns is a subspace in $\mathbb{R}^n$;

(d) The solution set of any homogeneous system of m equations in n unknowns is a subspace in $\mathbb{R}^n$.

Exercise 7.1. True or false (explain your answers):

(a) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0 (think about row equivalence and echelon form);

(b) Elementary row operations preserve rank;

(c) The rank of a matrix is equal to the maximum number of linearly independent columns one can find in the matrix;

(d) The rank of an $n \times n$ matrix is at most n;

(e) An $n \times n$ matrix having rank n is invertible.

Exercise 7.3. Compute the rank and find bases of all four fundamental subspaces for the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 7.4. Prove that if $A : X \to Y$ and V is a subspace of X then $\dim AV \le \operatorname{rank} A$. ($AV$ here means the subspace V transformed by the transformation A, i.e. any vector in AV can be represented as $A\mathbf{v}$, $\mathbf{v} \in V$). Deduce from here that $\operatorname{rank}(AB) \le \operatorname{rank} A$, where B is another linear map. **Hint:** Use the fact that the dimension of a subspace is no greater than the dimension of the space it is contained in.

Exercise 7.5. Prove that if $A : X \to Y$ and V is a subspace of X then $\dim AV \le \dim V$. Deduce from here that $\operatorname{rank}(AB) \le \dim B$, where B is another linear map.

Exercise 7.6 Prove that if the product AB of two $n \times n$ matrices is invertible, then both A and B are invertible. Avoid using determinants. **Hint:** Use the previous two problems.

Exercise 7.15 Complete the vectors $(1, 2, -1, 2, 3)^T$, $(2, 2, 1, 5, 5)^T$, $(-1, -4, 4, 7, -11)^T$ to a basis in $\mathbb{R}^5$.