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Ask questions!

Don't be afraid!

Suppose we have a column of 3 numbers

$$\text{e.g.}\quad \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1/2\\3\end{pmatrix}, \begin{pmatrix}\sqrt{2}\\2\\e\end{pmatrix}$$

Can view these as objects that can be <u>added</u>

$$\text{e.g.}\quad \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\1/2\\3\end{pmatrix} = \begin{pmatrix}2\\1/2\\3\end{pmatrix}$$

Can also <u>scale</u> them by a single number

$$\text{e.g.}\quad 5\begin{pmatrix}1\\1/2\\3\end{pmatrix} = \begin{pmatrix}5\\5/2\\ \end{pmatrix} \qquad \text{ties}$$

$$\frac{1}{2}\begin{pmatrix}\sqrt{2}\\2\\e\end{pmatrix} = \begin{pmatrix}\sqrt{2}/2\\1\\e/2\end{pmatrix}$$

Notice that addition and scaling satisfy certain expected properties

e.g. 1. commutative

$$\begin{pmatrix}1\\1/2\\1/3\end{pmatrix} + \begin{pmatrix}\sqrt{2}\\2\\e\end{pmatrix} = \begin{pmatrix}\sqrt{2}\\2\\e\end{pmatrix} + \begin{pmatrix}1\\1/2\\1/3\end{pmatrix}$$

2. associative

$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\1/2\\3\end{pmatrix}\right) + \begin{pmatrix}\sqrt{2}\\2\\e\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \left(\begin{pmatrix}1\\1/2\\3\end{pmatrix} + \begin{pmatrix}\sqrt{2}\\2\\e\end{pmatrix}\right)$$

3. neutral object or "zero"

$$\begin{pmatrix} 1 \\ 1/2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 3 \end{pmatrix}$$

4. objects have inverses ("negatives")

$$\begin{pmatrix} \sqrt{2} \\ 1 \\ e \end{pmatrix} + \begin{pmatrix} -\sqrt{2} \\ -1 \\ -e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

5. scaling by 1 changes nothing

$$1 \begin{pmatrix} 5 \\ 2 \\ \pi \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ \pi \end{pmatrix}$$

6. scaling is associative

$$(3 \cdot \pi) \begin{pmatrix} 1 \\ 1/2 \\ 3 \end{pmatrix} = 3 \left(\pi \begin{pmatrix} 1 \\ 1/2 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} 3\pi \\ 3\pi/2 \\ 9\pi \end{pmatrix}$$

7. scaling distributes pt. 1 e.g.

$$3 \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right) = 3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

8. scaling distributes pt. 2

$$= \begin{pmatrix} 9 \\ 9 \\ 15 \end{pmatrix}$$

$$(3+2) \begin{pmatrix} \sqrt{2} \\ 1 \\ \pi \end{pmatrix} = 3 \begin{pmatrix} \sqrt{2} \\ 1 \\ \pi \end{pmatrix} + 2 \begin{pmatrix} \sqrt{2} \\ 1 \\ \pi \end{pmatrix}$$

Vector Spaces

A vector space V is a collection of objects (called <u>vectors</u>, denoted in bold letters or as $\vec{v}, \vec{w}$, etc) w/ scaling and addition such that
usually by numbers in $\mathbb{R}$

$\alpha, \beta \in \mathbb{R}$
$\vec{u}, \vec{v}, \vec{w} \in V$

1) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

3) there exists a special element $\vec{0} \in V$ s.t. $\vec{v} + 0 = \vec{v}$ for any $\vec{v} \in V$.

4) For every $\vec{v} \in V$ there is a $\vec{w} \in V$ s.t. $\vec{v} + \vec{w} = 0$. Usually denote this $\vec{w}$ by $-\vec{v}$

5) $1\vec{v} = \vec{v}$ for all $v \in V$

6) $(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$

7) $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$

8) $(\alpha + \beta)\vec{v}$

If scalars come from $\mathbb{R}$, say V is a real vector space. Can also have complex vector spaces.

Ex: $\mathbb{R}^n$ = collection of all columns of size n

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$\alpha v = \alpha \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

Ex: $\mathbb{P}_n$ = polynomials of degree at most n

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

can scale and add polynomials

Ex: $m \times n$ matrix: an array of numbers, m rows, n cols

$$A = \left(a_{j,k}\right)_{j=1,\,k=1}^{m\quad n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

The transpose of A, denoted A^T, is formed by transforming the rows into cols

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ rows become cols, cols become rows

$$(A^T)_{j,k} = A_{k,j}$$

$$(1, 2, 3)^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Q: Prove that $\vec{0} \in V$ is unique.

Suppose $\hat{0}$ also satisfies (3).

Then $\vec{0} = \vec{0} + \hat{0}$ by property (3).

But $\underbrace{\vec{0} + \hat{0} = \hat{0} + \vec{0}}_{\text{commute}} = \hat{0}$ property (3).

So $\vec{0} = \hat{0}$.

Q: Show $0\vec{v} = \vec{0}$ $\forall v \in V$.

$$\underbrace{(0+0)\vec{v}}_{= 0\vec{v}} = 0\vec{v} + 0\vec{v} \underset{\text{inverses}}{\Longrightarrow} -(0\vec{v}) + 0\vec{v} = -(0\vec{v}) + 0\vec{v} + 0\vec{v}$$

$$\Downarrow$$

$$\vec{0} = \vec{0} + 0\vec{v}$$

$$\Rightarrow \vec{0} = 0\vec{v}$$

Q: Prove that additive inverses are unique.

Let $\vec{v}, \vec{w} \in V$. Suppose

$$\vec{v} + \vec{w} = 0.$$

Suppose $\vec{u}$ also satisfies $\vec{v} + \vec{u} = 0$.

$$\vec{v} + \vec{w} = \vec{v} + \vec{u}$$

Commutativity $\Rightarrow$ $\vec{w} + \vec{v} = 0$.

$$(\vec{w} + \vec{v}) + \vec{w} = (\vec{w} + \vec{v}) + \vec{u}$$

$$\vec{0} + \vec{w} = \vec{0} + \vec{u}$$

$$\Rightarrow \vec{w} = \vec{u}.$$

Q: V vector space. $\vec{v} \in V$.

Show that the additive inverse $-\vec{v}$ is the same as $(-1)\vec{v}$.

We are given $\vec{v} + (-\vec{v}) = 0$

we know $1\vec{v} = \vec{v}$

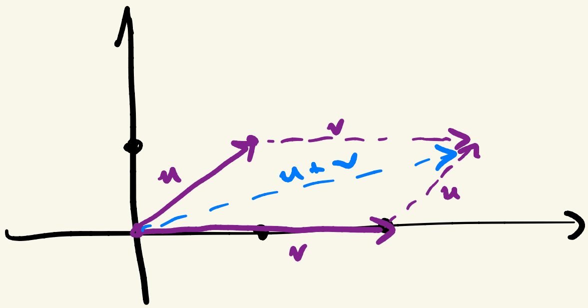
$$1\vec{v} + (-\vec{v}) = 0$$

OTOH, $1\vec{v} + (-1)\vec{v} = (1 + (-1))\vec{v} = 0\vec{v} = 0.$

Uniqueness $\Rightarrow$ $(-1)\vec{v} = -\vec{v}$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, vector addition and scaling can be interpreted geometrically.

vectors $\vec{u}, \vec{v}$ can be represented as arrows



$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$v = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$u + v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

§1.2 Linear combinations, bases.

V vector space. $\vec{v}_1, \dots, \vec{v}_p \in V$, $\alpha_1, \dots, \alpha_p \in \mathbb{R}$ (or $\mathbb{C}$)

A <u>linear combination</u> is a sum of the form

$$\alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p = \sum_{k=1}^{p} \alpha_k \vec{v}_k$$

<u>Definition</u>: A system/list/collection of vectors $\vec{v}_1, \dots, \vec{v}_n \in V$ is called a <u>basis</u> for V if <u>every</u> $v \in V$ can be <u>uniquely</u> written as

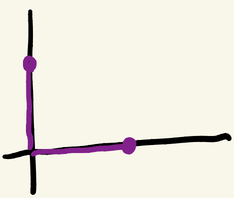
$$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$$

(i.e. there is exactly one choice of $\alpha_1, \dots, \alpha_n$)

- $\alpha_1, \dots, \alpha_n$ are called <u>coordinates</u> of $\vec{v}$ w.r.t. $\vec{v}_1, \dots, \vec{v}_n$

<u>e.g.</u> Let $V = \mathbb{R}^2$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a basis.



$$\begin{pmatrix} 5 \\ \pi \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \pi \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

more generally

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

non-examples

- $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$ is not a basis for $\mathbb{R}^2$

uniqueness fails

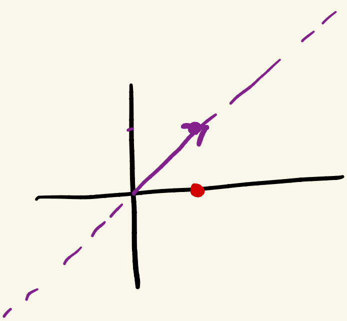
$$\begin{pmatrix}0\\0\end{pmatrix} = 0\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\end{pmatrix} + 0\begin{pmatrix}1\\1\end{pmatrix}$$

$$= 1\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} + (-1)\begin{pmatrix}1\\1\end{pmatrix}$$

two different sets of coefficients can both give 0 vector

- The list consisting of just $\begin{pmatrix}1\\1\end{pmatrix}$ is not a basis

e.g. $\begin{pmatrix}1\\0\end{pmatrix}$ cannot be expressed as a multiple of $\begin{pmatrix}1\\1\end{pmatrix}$



$\mathbb{P}_n$ has a basis

$$\vec{e}_0 := 1,\ \vec{e}_1 := t,\ \dots,\ \vec{e}_n := t^n$$

$$p(t) = a_0 + a_1 t + \dots + a_n t^n$$

$$\rightsquigarrow p(t) = a_0 \vec{e}_0 + a_1 \vec{e}_1 + \dots + a_n \vec{e}_n$$

The coordinates of $p(t)$ w.r.t. $\{e_i\}$ are $(a_0, a_1, \dots, a_n)$

Generating sets, linear independence

"every" "unique"

<u>Def</u>: $v_1, \dots, v_p \in V$ is called a spanning/generating set for V if any $\vec{v} \in V$ can be expressed

$$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p$$

for some $\alpha_1, \dots, \alpha_p \in \mathbb{F}$.

- Any basis is a spanning set.
- Adding extra vectors to any basis also yields a spanning set. why?

Def: A set of vectors $v_1, \ldots, v_p \in V$ is called <u>linearly independent</u> if the only linear combination giving $\vec{0}$ is the trivial one. i.e. the only solution to

$$\vec{0} = \alpha_1 \vec{v}_1 + \ldots + \alpha_p \vec{v}_p$$

is to set $\alpha_1 = \ldots = \alpha_p = 0$.

Otherwise, $\vec{v}_1, \ldots, \vec{v}_p$ are called <u>linearly dependent.</u>

So $\vec{v}_1, \ldots, \vec{v}_p$ are linearly dependent if and only if there are $\alpha_1, \ldots, \alpha_p$ not all 0 s.t.

$$\sum_{k=1}^{p} \alpha_k \vec{v}_k = \vec{0}.$$

<u>Aside: "if and only if"</u>

$P \Rightarrow Q$: "P implies Q", "whenever P is true, so is Q"
"P is sufficient for Q", "Q is necessary for P"
"P only if Q" "Q if P"

P: It is raining
Q: There are clouds in the sky

P if and only if Q means $P \Rightarrow Q$ and
$Q \Rightarrow P$

they are <u>logically equivalent</u>.

<u>Prop</u>: $v_1, \dots, v_p \in V$ lin dependent if and only if some v_k ($k \in \{1, \dots, p\}$) is a linear combination of the others,

$$v_k = \sum_{\substack{j=1 \\ j \neq k}}^{p} \beta_j v_j. \quad (*)$$

<u>Pf</u>: Suppose $v_1, \dots, v_p$ lin dependent. Then there are scalars $\alpha_1, \dots, \alpha_k$ not all 0 s.t.

$$\alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p = \vec{0}$$

$$\alpha_k \vec{v}_k = -\sum_{\substack{j=1 \\ j \neq k}}^{p} \alpha_j \vec{v}_j.$$

Then divide by α_k. (scale by $\frac{1}{\alpha_k}$)

Now suppose $(*)$ holds for some β_j's.

Then $$v_k - \sum_{\substack{j=1 \\ j \neq k}}^{p} \beta_j v_j = \vec{0}. \quad \square$$

Obs: Any basis is a linearly independent set.

why?

Prop: A system $\vec{v}_1, \dots, \vec{v}_n \in V$ is a basis if and only if lin indp. and spanning.

Pf: ($\Longrightarrow$) already established

($\Longleftarrow$) Suppose $\vec{v}_1, \dots, \vec{v}_n$ are lin indp and span V.

Let $v \in V$. v_i's span V, so there are some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ s.t.

$$v = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \sum_{k=1}^{n} a_k \vec{v}_k$$

Need to show that these α_i are unique.

Suppose $v = \tilde{\alpha}_1 \vec{v}_1 + \dots + \tilde{\alpha}_n \vec{v}_n$

Then $0 = (\alpha_1 - \tilde{\alpha}_1)\vec{v}_1 + \dots + (\alpha_n - \tilde{\alpha}_n)\vec{v}_n$.

Lin indp $\Longrightarrow$ $\alpha_1 - \tilde{\alpha}_1 = 0, \dots, \alpha_n - \tilde{\alpha}_n = 0$. $\square$

<u>Prop:</u> Any (finite) spanning set contains a basis.

<u>Pf:</u> Suppose $\vec{v}_1, \dots, \vec{v}_p$ spans V. If already lin indp, done. Otherwise, the list is linearly dependent. So some $\vec{v}_k$ is a linear combination of the others. Delete v_k. If new list is lin indp, good. Otherwise, do this again. Eventually, this process has to stop. $\square$